ON NON-NOETHERIAN IWASAWA THEORY

DAVID BURNS, ALEXANDRE DAOUD AND DINGLI LIANG

Abstract. We prove a general structure theorem for finitely-presented torsion modules over a class of commutative rings that need not be Noetherian. We then use this result to study the Weil-étale cohomology groups of $\mathbb{G}_m$ for curves over finite fields.

1. Introduction

Let p be a prime, k the function field of a smooth projective curve over the field with p elements and K a Galois extension of k for which $\mathrm{Gal}(K/k)$ is topologically isomorphic to the direct product $\mathbb{Z}_p^{\mathbb{N}}$ of a countably infinite number of copies of $\mathbb{Z}_p$. Then, since the completed p-adic group ring $\mathbb{Z}_p[[\mathbb{Z}_p^{\mathbb{N}}]]$ of $\mathbb{Z}_p^{\mathbb{N}}$ is not Noetherian, classical techniques of Iwasawa theory do not apply in this setting. With this problem in mind, Bandini, Bars and Longhi introduced a notion of 'pro-characteristic ideal' as a generalisation of the classical Iwasawa-theoretic notion of characteristic ideal, and used it to study several natural Iwasawa modules over K/k (cf. [2, 3, 4]). These efforts culminated in their proof, with Anglès, of a main conjecture for divisor class groups over Carlitz-Hayes cyclotomic extensions of k (see [1]) and, more recently, both Bandini and Coscelli [5] and Bley and Popescu [6] have extended this sort of result to a wider class of Drinfeld modular towers.

By adopting a more conceptual algebraic approach, we shall now strengthen the theory developed in these earlier articles. As the starting point for this, we identify a natural class of commutative rings (that includes, as a special case, all rings of the form $\mathbb{Z}_p[[\mathbb{Z}_p^{\mathbb{N}} \times G]]$ with G a finite abelian group) that are not, in general, Noetherian, but for which one can prove a structure theorem for a general category of finitely-presented, torsion modules (see Theorem 2.3). This result is of independent interest and, in particular, leads naturally to generalised notions of characteristic ideal that extend and refine the pro-characteristic ideal construction used previously.

We next prove that the inverse limits with respect to corestriction of the p-completions of the degree one Weil-étale cohomology groups of $\mathbb{G}_m$ over finite extensions of k in K are finitely-presented torsion $\mathbb{Z}_p[[\mathbb{Z}_p^{\mathbb{N}}]]$-modules. By applying our structure theory to these modules, we are then able to derive stronger, and more general, versions of the main results of the articles [1], [5] and [6] (see Theorem 3.7 and Remarks 3.10 and 3.11). At the same time, this approach also allows us to deduce that, somewhat surprisingly, outside a very restricted class of extensions K/k the inverse limit with respect to norms of the p-parts of the degree zero divisor class groups of finite extensions of k in K is not finitely generated as a $\mathbb{Z}_p[[\mathbb{Z}_p^{\mathbb{N}}]]$-module (see Corollary 3.8).

Finally, we note that there are natural families of Galois extensions of group $\mathbb{Z}_p^{\mathbb{N}}$ in number field settings (see, for example, the 'cyclotomic radical p-extensions' described by Mináč et al in [25, Th. A.1]), and that the algebraic results presented here can also in principle be used to study Iwasawa-theoretic modules over such extensions.

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2. Structure theories over non-Noetherian rings

In this section we fix a commutative unital ring A and write $Q(A)$ for its total quotient ring. We also write $\mathrm{ht}(\mathfrak{p})$ for the height of each $\mathfrak{p}$ in $\mathrm{Spec}(A)$ and consider the sets

$$\mathcal{P} = \mathcal{P}_A := \{\mathfrak{p} \in \mathrm{Spec}(A) : \mathrm{ht}(\mathfrak{p}) = 1\} \quad \text{and} \quad \mathcal{P}^{\mathrm{fg}} = \mathcal{P}_A^{\mathrm{fg}} := \{\mathfrak{p} \in \mathcal{P} : \mathfrak{p} \text{ is finitely generated}\}.$$

Given an A-module M, we write $M_\mathfrak{p}$ for its localisation at $\mathfrak{p}$ in $\mathrm{Spec}(A)$. We also write $M_{\mathrm{tor}} = M_{A\text{-}\mathrm{tor}}$ for the A-submodule of M comprising all elements m that are annihilated by a non-zero divisor of A (that may depend on m) and refer to M as a 'torsion A-module' if $M = M_{\mathrm{tor}}$ (or, equivalently, $Q(A) \otimes_A M = (0)$). We then define a (possibly empty) subset of $\mathcal{P}$ by setting

$$\mathcal{P}(M) = \mathcal{P}_A(M) := \mathcal{P} \cap \mathrm{Support}(M_{\mathrm{tor}}) = \{\mathfrak{p} \in \mathcal{P} : (M_{\mathrm{tor}})_\mathfrak{p} \neq (0)\}.$$

Finally, we write M_{tf} for the quotient of M by M_{tor}.

2.1. **Finitely-presented modules.**

2.1.1. *The general case.* The following notion will play a key role in the sequel.

Definition 2.1. A finitely generated A-module M will be said to be *admissible* if it has both of the following properties:

(P_1) for every $\mathfrak{p} \in \mathrm{Spec}(A)$ that is maximal amongst those contained in $\bigcup_{\mathfrak{q} \in \mathcal{P}(M)} \mathfrak{q}$, the localisation $A_\mathfrak{p}$ is a valuation ring (that is, its ideals are totally ordered by inclusion).

(P_2) $\mathcal{P}(M)$ is a finite subset of $\mathcal{P}^{\mathrm{fg}}$.

Remark 2.2.
(i) If $\mathcal{P}(M)$ is finite (as required by (P_2) and automatic if A is Noetherian), then the prime avoidance lemma implies (P_1) is valid if and only if $A_\mathfrak{q}$ is a valuation ring for every $\mathfrak{q}$ in $\mathcal{P}(M)$. In particular, if $A_\mathfrak{p}$ is a valuation ring for all $\mathfrak{p}$ in $\mathcal{P}$ (as is the case if A is either a Krull domain or valuation domain of arbitrary dimension), then M is admissible if and only if M_{tor} is supported on only finitely many primes in $\mathcal{P}$, each of which is finitely generated.
(ii) Prime ideals that are contained in a union of primes in $\mathcal{P}$ need not have height one. For example, if A is a Noetherian ring of dimension two, then Krull's Principal Ideal Theorem implies that every prime ideal of A is contained in $\bigcup_{\mathfrak{p} \in \mathcal{P}} \mathfrak{p}$.

As usual, an A-module will be said to be *pseudo-null* if its localisation vanishes at every prime in $\mathcal{P}$ and a map of A-modules will be said to be a *pseudo-isomorphism* if its kernel and cokernel are both pseudo-null.

We can now prove the structure result that is the starting point of our theory.

Theorem 2.3. *Let M be a finitely-presented A-module with property (P_1). Then all of the following claims are valid.*

(i) *If M is torsion, then there exists an A-module N, a finite family of principal ideals $\{ L_\tau \}_{\tau\in\mathcal{T}}$ of A and a pseudo-isomorphism of A-modules*

$$M \oplus N \to \bigoplus\nolimits_{\tau\in\mathcal{T}} A/L_\tau. \tag{1}$$

(ii) *If $Q(A)$ is semisimple, then the following claims are also valid.*
  (a) *There exists a pseudo-isomorphism of A-modules $M \to M_{\mathrm{tor}} \oplus M_{\mathrm{tf}}$.*
  (b) *Assume M is both admissible and torsion. Then in the pseudo-isomorphism (1) one can take the module N to be (0). Further, there exists a finite index set $\mathcal{S}$ and for each $\sigma \in \mathcal{S}$ a prime ideal $\mathfrak{p}_\sigma$ in $\mathcal{P}$ and a natural number a_σ, for which there exists a pseudo-isomorphism of A-modules $M \to \bigoplus_{\sigma\in\mathcal{S}} A/\mathfrak{p}_\sigma^{a_\sigma}$.*

Proof. To prove (i) we assume that M is A-torsion. We also note that if $\mathcal{P}(M) = \emptyset$, then M is pseudo-null and there is nothing to prove. We therefore assume that $\mathcal{P}(M) \neq \emptyset$, set $S := A \setminus \bigcup_{\mathfrak{p}\in\mathcal{P}(M)} \mathfrak{p}$ and write $(-)'$ for the localisation functor $S^{-1}(-)$.

The maximal ideals of A' are in one-to-one correspondence with the primes of A that are maximal amongst those contained in $\bigcup_{\mathfrak{q}\in\mathcal{P}(M)} \mathfrak{q}$. Hence, from condition (P_1), it follows that the localisation of A' at each maximal ideal is a valuation ring. We may therefore apply Warfield's Structure Theorem [30, Th. 3] to deduce the existence of an A'-module N' and a finite collection $\{a'_\tau\}_{\tau\in\mathcal{T}}$ of elements of $A' \setminus (A')^\times$ for which there is an isomorphism of A'-modules

$$\psi : M' \oplus N' \cong \bigoplus\nolimits_{\tau\in\mathcal{T}} A'/(a'_\tau). \tag{2}$$

We now choose elements $\{a_\tau\}_{\tau\in\mathcal{T}}$ of $A \setminus S = \bigcup_{\mathfrak{p}\in\mathcal{P}(M)} \mathfrak{p}$ with $(a_\tau)' = (a'_\tau)$ for each $\tau \in \mathcal{T}$. Then, since both M and $\bigoplus_{\tau\in\mathcal{T}} A/(a_\tau)$ are finitely-presented A-modules (the former by assumption and the latter clearly), the canonical maps

$$\begin{aligned} &\mathrm{Hom}_A\big(M, \bigoplus\nolimits_{\tau\in\mathcal{T}} A/(a_\tau)\big)' \xrightarrow{\sim} \mathrm{Hom}_{A'}\big(M', \bigoplus\nolimits_{\tau\in\mathcal{T}} A'/(a'_\tau)\big), \\ &\mathrm{Hom}_A\big(\bigoplus\nolimits_{\tau\in\mathcal{T}} A/(a_\tau), M\big)' \xrightarrow{\sim} \mathrm{Hom}_{A'}\big(\bigoplus\nolimits_{\tau\in\mathcal{T}} A'/(a'_\tau), M'\big) \end{aligned} \tag{3}$$

are both bijective. This implies the existence of homomorphisms of A-modules

$$\iota_1 : M \to \bigoplus\nolimits_{\tau\in\mathcal{T}} A/(a_\tau) \quad \text{and} \quad \iota_2 : \bigoplus\nolimits_{\tau\in\mathcal{T}} A/(a_\tau) \to M$$

such that, for suitable elements s_1 and s_2 of S, the maps ι'_1/s_1 and ι'_2/s_2 are respectively equal to the composites

$$M' \xrightarrow{(\mathrm{id},0)} M' \oplus N' \xrightarrow{\psi} \bigoplus\nolimits_{\tau\in\mathcal{T}} A'/(a'_\tau) \quad \text{and} \quad \bigoplus\nolimits_{\tau\in\mathcal{T}} A'/(a'_\tau) \xrightarrow{\psi^{-1}} M' \oplus N' \xrightarrow{(\mathrm{id},0)} M'.$$

Set $N := \ker(\iota_2)$. Then, since the endomorphism $\iota_2 \circ \iota_1$ of M is given by multiplication by $s_2 s_1$ and the latter element is not contained in any prime in $\mathcal{P}(M)$, the modules $\ker(\iota_1)$, $\mathrm{coker}(\iota_2)$ and $\iota_1(M) \cap N$ are all pseudo-null. In addition, by localising the exact commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r, "\subseteq"] \arrow[d, equal] & \iota_1(M)+N \arrow[r, "\iota_2"] \arrow[d, "\subseteq"'] & (\iota_2\circ\iota_1)(M) \arrow[r] \arrow[d, "\subseteq"'] & 0 \\
0 \arrow[r] & N \arrow[r, "\subseteq"] & \bigoplus_{\tau\in\mathcal{T}} A/(a_\tau) \arrow[r, "\iota_2"] & \mathrm{im}(\iota_2) \arrow[r] & 0
\end{tikzcd}$$

one checks that the inclusion

$$\iota_1(M)+N \to \bigoplus\nolimits_{\tau\in\mathcal{T}} A/(a_\tau)$$

is also a pseudo-isomorphism. Given these facts, the tautological short exact sequence

$$0 \to \iota_1(M)\cap N \xrightarrow{x\mapsto(x,x)} \iota_1(M)\oplus N \xrightarrow{(x,y)\mapsto x-y} \iota_1(M)+N \to 0$$

implies that the composite map

$$M\oplus N \xrightarrow{(\iota_1,\mathrm{id})} \iota_1(M)\oplus N \xrightarrow{(x,y)\mapsto x-y} \bigoplus\nolimits_{\tau\in\mathcal{T}} A/(a_\tau) \tag{4}$$

is a pseudo-isomorphism. This proves (i) with $L_\tau = (a_\tau)$ for each $\tau\in\mathcal{T}$.

In the remainder of the argument we no longer require, except when explicitly stated, that M is a torsion module, but we do assume that the ring $Q(A)$ is semisimple, and hence von Neumann regular. In particular, since the localisation of A' at each maximal ideal is a valuation ring, results of Endo [14, §5, Prop. 10, Prop. 11 and Cor.] then imply the existence of a direct product decomposition

$$A' = \prod\nolimits_{t\in T} A'_t$$

over a finite index set T in which each ring A'_t is a semi-hereditary (or Prüfer) domain.

In particular, if M is an admissible, torsion module, then $\mathcal{P}(M)$ is finite and, for each $t\in T$, the ring A'_t is a semi-local Prüfer domain and the A'_t-component of M' is both finitely-presented and torsion. In this case, therefore, we can apply the stronger structure theorem of Fuchs and Salce [17, Cor. III.6.6, Th. V.3.4] to each ring A'_t in order to deduce the existence of an isomorphism (2) for which the module N' is zero. Then, in this case, the module $\mathrm{coker}(\iota_1)' = \mathrm{coker}(\psi)$ vanishes and so $\mathrm{coker}(\iota_1)_{\mathfrak{p}}$, and hence also $N_{\mathfrak{p}}$, vanishes for all $\mathfrak{p}$ in $\mathcal{P}(M)$.

Next we suppose, in addition, that every prime ideal in $\mathcal{P}(M)$ is finitely generated and we claim this implies that every prime ideal of A' is finitely generated. To see this we note every prime ideal of A' is of the form $\mathfrak{B} = \mathfrak{B}_0\times\prod_{t\in T\setminus\{t_0\}} A'_t$ where $\mathfrak{B}_0$ is a prime ideal of the domain A'_{t_0} for some $t_0\in T$. If $\mathfrak{B}_0 = (0)$, then $\mathfrak{B}$ is finitely generated. If $\mathfrak{B}_0 \neq (0)$, then $\mathfrak{Q} := (0)\times\prod_{t\in T\setminus\{t_0\}} A'_t$ is a prime ideal of A' that is strictly contained in $\mathfrak{B}$. Now, since $\mathcal{P}(M)$ is assumed to be finite, the prime avoidance lemma implies that $\mathfrak{B}$ and $\mathfrak{Q}$ correspond to prime ideals $\mathfrak{p}$ and $\mathfrak{p}_1$ of A with $\mathfrak{p}_1 \subsetneq \mathfrak{p} \subseteq \mathfrak{q}$ for some $\mathfrak{q}\in\mathcal{P}(M)$. In particular, since $\mathfrak{q}$ has height one, this implies $\mathfrak{p} = \mathfrak{q}$ and hence that $\mathfrak{B}$ is finitely generated, as claimed.

At this stage, we can apply Cohen's Theorem [11, Th. 2] to deduce that A', and hence each of its components A'_t, is Noetherian. It follows that the localisation $A'_{\mathfrak{B}}$ of A' at each prime ideal $\mathfrak{B}$ is Noetherian, a domain (as each component A'_t of A' is a domain) and either a field (if $\mathfrak{B}$ corresponds to the zero ideal of some component A'_t) or a valuation ring (by Remark 2.2 and the assumption M is admissible). We further recall that every

Noetherian valuation ring that is not a field is a discrete valuation ring (cf. [22, Th. 5.18]). Taken together, these facts imply that every component ring A'_t of A' is a Dedekind domain. We can therefore now appeal to the usual structure theorem for finitely generated torsion modules over such rings to deduce that the isomorphism (2) can be replaced by an isomorphism of the form $M' \cong \bigoplus_{\sigma\in\mathcal{S}} A'/(\mathfrak{p}_\sigma^{a_\sigma})'$ in which $\mathcal{S}$ is a finite index set, each $\mathfrak{p}_\sigma$ a prime ideal in $\mathcal{P}(M)$ and each a_σ a natural number. There are then also associated isomorphisms (3) in which $\mathcal{T}$ is replaced by $\mathcal{S}$ and each of the terms (a_τ) and (a'_τ) by $\mathfrak{p}_\tau^{a_\tau}$ and $(\mathfrak{p}_\tau^{a_\tau})'$ respectively, and so one can deduce the existence of corresponding analogues of the homomorphisms ι_1 and ι_2. In addition, in this case the module $N := \ker(\iota_2)$ is pseudo-null (since $N' = (0)$ and we already observed that $N_\mathfrak{p}$ vanishes for all $\mathfrak{p}$ in $\mathcal{P}(M)$) and so can be taken to be zero in the pseudo-isomorphism that arises from the analogue of the construction (4) in this case. This proves (ii)(b).

Finally, to prove (ii)(a), we do not assume either that M is torsion or that $M_{\rm tor}$ is admissible. We do however continue to assume that $Q(A)$ is semisimple and hence, by the above argument, that A' is a finite direct product of semi-hereditary domains. Thus, by the general result of [14, §5, Cor.], we know that $M'_{\rm tf}$ is a projective A'-module and hence that there exists an isomorphism of A'-modules of the form $M' \cong M'_{\rm tf} \oplus M'_{\rm tor}$.

Now, since M is a finitely-presented A-module, the natural map

$$\mathrm{Hom}_A(M, M_{\rm tor})' \to \mathrm{Hom}_{A'}(M', M'_{\rm tor})$$

is bijective. In particular, there exists a homomorphism $\phi : M \to M_{\rm tor}$ and an element $s_1 \in S$ with the property that ϕ'/s_1 corresponds under this identification to the projector of M' onto $M'_{\rm tor}$. As such, ϕ'/s_1 restricts to the submodule $M'_{\rm tor}$ to give the identity. We can therefore find an element s_2 of S such that the map $\tau := s_2 \cdot \phi$ restricted to $M_{\rm tor}$ is equal to $s_1s_2 \cdot \mathrm{id}_{M_{\rm tor}}$.

We now write π for the canonical projection $M \to M_{\rm tf}$ and consider the map

$$\kappa : M \to M_{\rm tf} \oplus M_{\rm tor}; \quad m \mapsto (\pi(m), \tau(m)).$$

One then checks that $\ker(\kappa) = \ker(\tau) \cap M_{\rm tor}$ and that $\mathrm{coker}(\kappa)$ is equal to the cokernel of the endomorphism of $M_{\rm tor}$ induced by τ and, since $s_1s_2 \in S$, these modules are both pseudo-null. It follows that the above map κ is the required pseudo-isomorphism. $\square$

In view of Theorem 2.3(ii), the following class of rings will be of interest to us in the sequel.

Definition 2.4. A commutative unital ring A will be said to be *admissible* if it has both of the following properties:

(P_3) $Q(A)$ is semisimple.
(P_4) Every finitely-presented torsion A-module is admissible (as in Definition 2.1).

It is clear that a Noetherian integrally closed domain (or equivalently, a Noetherian Krull domain) is admissible in the above sense and also such that every finitely generated module is finitely-presented. For such rings, Theorem 2.3 simply recovers the classical structure theorem of Bourbaki [7, Chap. VII, § 4, Th. 4 and Th. 5]. However, Theorem 2.3 can also be applied in more general situations and, to end this section, we shall now discuss some examples that are relevant to later arguments.

Remark 2.5.
(i) Let A be an arbitrary Krull domain. Then $Q(A)$ is a field (and so semisimple), $\mathcal{P}_A$ is non-empty, the localisation of A at each prime in $\mathcal{P}_A$ is a discrete valuation ring and every non-zero ideal is contained in only finitely many primes in $\mathcal{P}_A$. Hence, if M is a non-zero finitely generated torsion A-module, then $\mathcal{P}_A(M)$ is finite (as it is the subset of $\mathcal{P}_A$ comprising primes containing the annihilator of M) and so M has property (P_1) (by Remark 2.2(i)) and also admits a pseudo-isomorphism (1) with $N = (0)$. In particular, A is admissible if $\mathcal{P}_A = \mathcal{P}_A^{\mathrm{fg}}$. However, there are Krull domains A for which $\mathcal{P}_A \neq \mathcal{P}_A^{\mathrm{fg}}$ (see, for instance, the examples discussed by Eakins and Heinzer in [13]) and no such ring is admissible. Indeed, in any such case, if $\mathfrak{p} \in \mathcal{P}_A$ is not finitely generated and $x \in \mathfrak{p} \setminus \{0\}$, then $M := A/(xA)$ is a finitely presented torsion A-module with $\mathfrak{p} \in \mathcal{P}_A(M)$.
(ii) If A is a unique factorisation domain, then A is a Krull domain for which every prime in $\mathcal{P}_A$ is principal and so the above discussion implies A is admissible. In fact, for such a ring, the only essential difference between the argument of Theorem 2.3 and that of Bourbaki referred to above is that we require the module M to be finitely-presented, rather than merely finitely generated, in order to guarantee the existence of the isomorphism (3).

2.1.2. *Group rings.* In this subsection we assume to be given a $\mathbb{Z}_p$-algebra R that is an integrally closed domain of characteristic zero. For a fixed finite abelian group G, we compare the notions of admissibility introduced above relative to R and to the group ring $A := R[G]$ of G over R.

To do this, we write f for the ring inclusion $R \to A$, $f^* : \mathrm{Spec}(A) \to \mathrm{Spec}(R)$ for the induced morphism of spectra and $f^*(M)$ for each A-module M for the R-module obtained by restriction through f. We note that A is a free R-module of finite rank (as G is finite) so that f is a finite, flat ring morphism. In addition, since $|G|$ is invertible in the field of fractions $Q(R)$ of R, the algebra $Q(A)$ is equal to $Q(R)[G]$ and is therefore a finite product $\prod_{i\in I} K_i$ of finite degree field extensions K_i of $Q(R)$ (and so is semisimple).

We write $D(n)$ for the set of positive divisors of a natural number n. We also fix a primitive n-th root of unity ζ_n in $\mathbb{Q}_p^c$, set $L_n := \mathbb{Q}_p(\zeta_n)$ and write $\mathcal{O}_n$ for its valuation ring $\mathbb{Z}_p[\zeta_n]$. We then set $R_n := R \otimes_{\mathbb{Z}_p} \mathcal{O}_n$ and write ι_n for the ring inclusion $R \to R_n$.

Proposition 2.6. *Fix $R, G, A = R[G]$ and f as above, and write H for the maximal subgroup of G of order prime to p. Then the following claims are valid.*

(i) *For $\mathfrak{q} \in \mathrm{Spec}(R)$, the fibre $(f^*)^{-1}(\mathfrak{q}) := \{\mathfrak{p} \in \mathrm{Spec}(A) : f^*(\mathfrak{p}) = \mathfrak{q}\}$ is finite and non-empty. For $\mathfrak{p} \in \mathrm{Spec}(A)$, one has $\mathrm{ht}(\mathfrak{p}) = \mathrm{ht}(f^*(\mathfrak{p}))$ and so $\mathfrak{p} \in \mathcal{P}_A \Longleftrightarrow f^*(\mathfrak{p}) \in \mathcal{P}_R$.*
(ii) *Fix $\mathfrak{q} \in \mathcal{P}_R$ and write $D_{\mathfrak{q}}(|G|)$ for $D(|G|)$ if $p \notin \mathfrak{q}$ and for $D(|H|)$ if $p \in \mathfrak{q}$.*
  (a) *$(f^*)^{-1}(\mathfrak{q}) \subseteq \mathcal{P}_A^{\mathrm{fg}} \Longleftrightarrow (\iota_n^*)^{-1}(\mathfrak{q}) \subseteq \mathcal{P}_{R_n}^{\mathrm{fg}}$ for every $n \in D_{\mathfrak{q}}(|G|)$.*
  (b) *Assume $R_{\mathfrak{q}}$ is a valuation ring. Then $A_{\mathfrak{p}}$ is a valuation ring for all $\mathfrak{p} \in (f^*)^{-1}(\mathfrak{q})$ if and only if both $|G| \notin \mathfrak{q}$ and $f^*(A)_{\mathfrak{q}}$ is a maximal $R_{\mathfrak{q}}$-order in $Q(A)$.*
(iii) *For any finitely generated A-module M the following equivalences are valid:*
  (a) *M is finitely-presented (over A) $\Longleftrightarrow$ $f^*(M)$ is finitely-presented (over R);*
  (b) *$f^*(M_{\mathrm{tor}})$ is the R-torsion submodule of $f^*(M)$. In particular, M is a torsion A-module $\Longleftrightarrow$ $f^*(M)$ is a torsion R-module;*
  (c) *$\mathcal{P}_A(M) \subseteq (f^*)^{-1}(\mathcal{P}_R(f^*(M)))$ and so $\mathcal{P}_A(M)$ is finite if $\mathcal{P}_R(f^*(M))$ is finite;*
  (d) *M is a pseudo-null A-module if $f^*(M)$ is a pseudo-null R-module.*

Proof. Since f is both finite and flat it has the lying over, incomparability and going down properties and, in addition, its fibres are finite (cf. [24, Chap. 3, Th. 9.3, Th. 9.5 and Exer. 9.3]). The first assertion of (i) is thus clear. For the second assertion, it is enough to show $\mathrm{ht}(\mathfrak{p}) = \mathrm{ht}(f^*(\mathfrak{p}))$ for $\mathfrak{p} \in \mathrm{Spec}(A)$. For this, we claim first that $\mathrm{ht}(\mathfrak{p}) \geq \mathrm{ht}(f^*(\mathfrak{p}))$: indeed, this follows easily from the fact that if $\{\mathfrak{b}', \mathfrak{b}\} \subset \mathrm{Spec}(R)$ and $\mathfrak{a} \in \mathrm{Spec}(A)$ are such that $\mathfrak{b}' \subsetneq \mathfrak{b}$ and $f^*(\mathfrak{a}) = \mathfrak{b}$, then (by going down) there exists $\mathfrak{a}' \in \mathrm{Spec}(A)$ with $\mathfrak{a}' \subsetneq \mathfrak{a}$ and $f^*(\mathfrak{a}') = \mathfrak{b}'$. On the other hand, one has $\mathrm{ht}(\mathfrak{p}) \leq \mathrm{ht}(f^*(\mathfrak{p}))$ since for any inclusion $\mathfrak{a}' \subsetneq \mathfrak{a}$ with $\mathfrak{a}'$ and $\mathfrak{a}$ in $\mathrm{Spec}(A)$, incomparability implies that the inclusion $f^*(\mathfrak{a}') \subset f^*(\mathfrak{a})$ is also strict. This proves (i).

We next make a general observation. For this, we fix a natural number m, a quotient Q of G, an ideal J of $\mathcal{O}_m[Q]$, set $R_m[Q]/J := R_m \otimes_{\mathcal{O}_m} (\mathcal{O}_m[Q]/J)$ and use the canonical ring homomorphisms $f_{m,J} : R_m \to R_m \otimes_{\mathcal{O}_m} (\mathcal{O}_m[Q]/J)$ and $f_m^J : R_m[Q] \to R_m[Q]/J$. We assume $J \cap \mathcal{O}_m = (0)$ (in $\mathcal{O}_m[Q]$) and $\mathcal{O}_m[Q]/J$ is $\mathcal{O}_m$-free and hence that $f_{m,J} \circ \iota_m$ is an injective finite flat ring morphism $R \to R_m[Q]/J$. Via this morphism, we regard $R_m[Q]/J$ as an extension of R and note the argument of (i) implies that any prime ideal of $R_m[Q]/J$ lying over $\mathfrak{q}$ has height one. In addition, since $\ker(f_m^J) = R_m \otimes_{\mathcal{O}_m} J$ is finitely generated (as $\mathcal{O}_m[Q]/J$ is $\mathcal{O}_m$-free and $\mathcal{O}_m[Q]$ is Noetherian), for each $\mathfrak{p} \in \mathcal{P}_{R_m[Q]/J}$ one has

$$\big(\mathfrak{p} \cap R = \mathfrak{q} \Longleftrightarrow (f_m^J)^{-1}(\mathfrak{p}) \cap R = \mathfrak{q}\big) \text{ and } \big(\mathfrak{p} \in \mathcal{P}^{\mathrm{fg}}_{R_m[Q]/J} \Longleftrightarrow (f_m^J)^{-1}(\mathfrak{p}) \in \mathcal{P}^{\mathrm{fg}}_{R_m[Q]}\big). \quad (5)$$

Turning now to the proof of (ii), we first note that, for each $n \in D(|G|)$, the morphism ι_n is finite and flat and so the argument of (i) implies $(\iota_n^*)^{-1}(\mathfrak{q}) \subseteq \mathcal{P}_{R_n}$. We next fix a homomorphism $\psi : G \to \mathbb{Q}_p^{c,\times}$ of exact order n. Then the kernel J_ψ of the induced $\mathbb{Z}_p$-linear ring homomorphism $\psi_* : \mathbb{Z}_p[G] \to \mathbb{Q}_p^c$ is such that $J_\psi \cap \mathbb{Z}_p = (0)$ and $\mathbb{Z}_p[G]/J_\psi \cong \mathrm{im}(\psi_*)$ is $\mathbb{Z}_p$-free (so that the criteria (5) are valid with $m = 1$, $Q = G$ and $J = J_\psi$). In particular, since the algebra $R[G]/J_\psi$ identifies with $R \otimes_{\mathbb{Z}_p} \mathrm{im}(\psi_*) = R_n$, this shows that the stated condition on the sets $(\iota_n^*)^{-1}(\mathfrak{q})$ in (ii)(a) are necessary.

To prove its sufficiency, we will show it implies, for every $m \in D(|G|)$ and every quotient Q of G, that each prime ideal of $R_m[Q]$ lying over $\mathfrak{q}$ is finitely generated. To prove this, we argue by induction on $|Q|$, with the case $|Q| = 1$ being obvious. To deal with the induction step, we fix $m \in D(|G|)$, a prime divisor ℓ of $|Q|$, a non-trivial element σ of Q that has ℓ-power order $t = \ell^d$ and is such that Q decomposes as a direct product $\langle\sigma\rangle \times Q'$ and a prime ideal $\mathfrak{p}$ of $R_m[Q]$ that lies over $\mathfrak{q}$. Now, if $\sigma^{t/\ell} - 1 \in \mathfrak{p}$, then $\mathfrak{p}$ is the full-preimage under the canonical projection $R_m[Q] \to R_m[Q/\langle\sigma^{t/\ell}\rangle]$ of a prime ideal and so, by induction (and an application of (5) with J the kernel of $\mathcal{O}_m[Q] \to \mathcal{O}_m[Q/\langle\sigma^{t/\ell}\rangle]$), is finitely generated. On the other hand, if $\sigma^{t/\ell} - 1 \notin \mathfrak{p}$ and we set $T_\sigma := \sum_{j=0}^{\ell-1} (\sigma^{t/\ell})^j$, then the equality $(\sigma^{t/\ell} - 1)T_\sigma = 0$ implies $T_\sigma \in \mathfrak{p}$. To deal with this case, we fix an injective homomorphism $\psi : \langle\sigma\rangle \to \mathcal{O}_t^\times$ and consider the induced (surjective) $\mathcal{O}_m$-linear ring homomorphism

$$\psi_{m,*} : \mathcal{O}_m[Q] = \mathbb{Z}_p[\langle\sigma\rangle] \otimes_{\mathbb{Z}_p} \mathcal{O}_m[Q'] \to \mathcal{O}_t \otimes_{\mathbb{Z}_p} \mathcal{O}_m[Q'] = (\mathcal{O}_t \otimes_{\mathbb{Z}_p} \mathcal{O}_m)[Q'] \cong \prod\nolimits_C \mathcal{O}_a[Q']$$

where $a = a(m,t) \in D(|G|)$ is the least common multiple of m and t and, with b denoting the greatest common divisor of m and t, we write C for a fixed set of coset representatives for $\mathrm{Gal}(L_a/L_b)$ in $\mathrm{Gal}(L_a/\mathbb{Q}_p)$, Then $\ker(\psi_{m,*}) = \mathcal{O}_m[Q] \cdot T_\sigma$ and so the containment $T_\sigma \in \mathfrak{p}$ implies $\mathfrak{p}$ is the full preimage under the projection $R_m \otimes_{\mathcal{O}_m} \psi_{m,*} : R_m[Q] \to \prod_C R_a[Q']$

of a prime ideal. Hence, by the induction hypothesis (and an application of (5) with $J = \ker(\psi_{m,*})$), it follows again that $\mathfrak{p}$ is finitely generated.

To complete the proof of (ii)(a) we now only need to show that if $G = H \times P$ with P a non-trivial p-group, then for any $\mathfrak{q} \in \mathcal{P}_R$ that contains p, one has $(f^*)^{-1}(\mathfrak{q}) \subseteq \mathcal{P}_A^{\mathrm{fg}}$ if $(\iota_n^*)^{-1}(\mathfrak{q}) \subseteq \mathcal{P}_{R_n}^{\mathrm{fg}}$ for all $n \in D(|H|)$. Now f factors as the composite $f_P \circ f_H$ of the finite, flat ring morphisms $f_H : R \to R[H]$ and $f_P : R[H] \to (R[H])[P] = A$ and by what we have just proved, the given condition implies that $(f_H^*)^{-1}(\mathfrak{q}) \subseteq \mathcal{P}_{R[H]}^{\mathrm{fg}}$. It is thus enough to note that if $\mathfrak{q}' \in (f_H^*)^{-1}(\mathfrak{q})$, then $p \in \mathfrak{q}'$ and so the only prime ideal in $(f_P^*)^{-1}(\mathfrak{q}')$ is $\mathfrak{q}' + I(P) \cdot R[G]$ which is finitely generated (over R) since $\mathfrak{q}'$ is.

Turning to (ii)(b) we assume $R_\mathfrak{q}$ is a valuation ring and note that, as R is a $\mathbb{Z}_p$-algebra, one has $|G| \in \mathfrak{q}$ if and only if both $p \in \mathfrak{q}$ and $p \mid |G|$. In particular, if this last condition is satisfied, then $(f^*)^{-1}(\mathfrak{q})$ contains the ideal $\mathfrak{p} = \mathfrak{q}' + I(P) \cdot R[G]$ discussed above. One then checks $A_\mathfrak{p}$ is equal to $(R[H])_{\mathfrak{q}'}[P]$ which is not an integral domain (as P is non-trivial) and so cannot be a valuation ring. To prove (ii)(b) it is thus enough to assume $|G| \notin \mathfrak{q}$ and show $A_\mathfrak{p}$ is a valuation ring for all $\mathfrak{p} \in \Sigma := (f^*)^{-1}(\mathfrak{q})$ if and only if $f^*(A)_\mathfrak{q}$ is a maximal $R_\mathfrak{q}$-order in $Q(A)$. In this case, there exist subrings $\mathcal{O}_i$ of K_i that are integral over $R_\mathfrak{q}$ and have K_i as their fraction field and are such that

$$f^*(A)_\mathfrak{q} = R_\mathfrak{q}[G] = \prod\nolimits_{i \in I} \mathcal{O}_i. \tag{6}$$

It follows that $f^*(A)_\mathfrak{q}$ is a maximal $R_\mathfrak{q}$-order if and only if each $\mathcal{O}_i$ is the integral closure $\mathcal{O}_i'$ of $R_\mathfrak{q}$ in K_i. In addition, writing $\Sigma(i)$ for the (finite) set of non-zero prime, and hence maximal, ideals of $\mathcal{O}_i$, the set $(f^*)^{-1}(\mathfrak{q})$ corresponds bijectively to $\bigcup_{i \in I} \Sigma(i)$ in the following way: for each $\mathfrak{p} \in \Sigma$, there exists a unique $i_\mathfrak{p} \in I$ and a unique $\mathfrak{P}_\mathfrak{p} \in \Sigma(i_\mathfrak{p})$ such that $A_\mathfrak{p} = \mathcal{O}_{i_\mathfrak{p}, \mathfrak{P}_\mathfrak{p}}$ (and $\mathfrak{P}_\mathfrak{p} \cap R = \mathfrak{q}$). In addition, by Chevalley's Extension Theorem, each ring $\mathcal{O}_i'$ is the intersection of the finitely many valuation subrings of K_i that extend $R_\mathfrak{q}$ and the localisation of $\mathcal{O}_i'$ at any of its maximal ideals is equal to one of these valuation rings (cf. [15, Lem. 3.2.6]).

We now assume $A_\mathfrak{p}$ is a valuation ring for every $\mathfrak{p} \in \Sigma$. In this case $\mathcal{O}_{i_\mathfrak{p}, \mathfrak{P}}$ is a valuation ring that extends $R_\mathfrak{q}$ for every $\mathfrak{P} \in \Sigma(i_\mathfrak{p})$ and hence, since $\mathcal{O}_{i_\mathfrak{p}} = \bigcap_{\mathfrak{P} \in \Sigma(i_\mathfrak{p})} (\mathcal{O}_{i_\mathfrak{p}})_\mathfrak{P}$ (as $\mathcal{O}_{i_\mathfrak{p}}$ is an integral domain), one must have $\mathcal{O}_{i_\mathfrak{p}}' \subseteq \mathcal{O}_{i_\mathfrak{p}}$ and therefore also $\mathcal{O}_{i_\mathfrak{p}} = \mathcal{O}_{i_\mathfrak{p}}'$. Thus, in this case, (6) implies that $f^*(A)_\mathfrak{q}$ is integrally closed in $Q(A)$ and so is a maximal $R_\mathfrak{q}$-order.

Conversely, if $f^*(A)_\mathfrak{q}$ is a maximal $R_\mathfrak{q}$-order, then (6) implies that $\mathcal{O}_i = \mathcal{O}_i'$ for all $i \in I$. In particular, since the localisation of each $\mathcal{O}_i'$ at any of its maximal ideals is a valuation ring that extends $R_\mathfrak{q}$, it follows that the localisation $\mathcal{O}_{i_\mathfrak{p}, \mathfrak{P}_\mathfrak{p}}'$ of A at each $\mathfrak{p} \in \Sigma$ is a valuation subring of some field K_i, as required to complete the proof of (ii).

The proof of (iii) relies crucially on the fact A is a free R-module of finite rank. In (iii)(a), the forward implication is clear and the reverse implication a consequence of Schanuel's Lemma. To prove (iii)(b) it is enough to prove the first assertion and then, since every non-zero element of R is a non-zero divisor of A, it is enough to show that any element m of M that is annihilated by a non-zero divisor a of A is also annihilated by a non-zero element of R. To prove this we write $f_a(X)$ for the monic polynomial of minimal degree in $R[X]$ with $f_a(a) = 0$ and note that the constant term of $f_a(X)$ is non-zero (since a is a non-zero divisor and $f_a(X)$ is chosen to be of minimal degree) and annihilates m. To prove (iii)(c), we note (iii)(b) implies $f^*(M_{\rm tor})$ is the R-torsion submodule of $f^*(M)$. We then fix $\mathfrak{p} \in \mathcal{P}_A(M)$ and an element m of $M_{\rm tor}$ with non-zero image in $M_{{\rm tor},\mathfrak{p}}$. Then $\mathfrak{p}$ contains the annihilator $\mathcal{A}(m)$ of m in A and so $f^*(\mathfrak{p})$ contains the annihilator $R \cap \mathcal{A}(m)$ of m in R. The image of m in $f^*(M_{\rm tor})_{f^*(\mathfrak{p})}$ is therefore non-zero so that $f^*(\mathfrak{p}) \in \mathcal{P}_R(f^*(M))$ and hence $\mathfrak{p} \in (f^*)^{-1}(\mathcal{P}_R(f^*(M))$, as required. Finally, (iii)(d) is true since (iii)(c) implies that $\mathcal{P}_A(M) = \emptyset$ if $\mathcal{P}_R(f^*(M)) = \emptyset$. $\square$

We now consider, for each natural number n, the following subset of $\mathrm{Spec}(R)$

$$\mathcal{P}_R^n := \{\mathfrak{q} \in \mathcal{P}_R : n \notin \mathfrak{q} \text{ and } (\iota_m^*)^{-1}(\mathfrak{q}) \subseteq \mathcal{P}_{R_m}^{\rm fg} \text{ for all } m \in D(n)\}.$$

Example 2.7. By taking $m = 1$ $(\in D(n))$ in the above definition, it is clear $\mathcal{P}_R^n \subseteq \mathcal{P}_R^{\rm fg}$. Under certain hypotheses on R, such as the following, it is possible to be much more precise.
(i) If R is Noetherian, then clearly $\mathcal{P}_R^n = \{\mathfrak{q} \in \mathcal{P}_R : p \notin \mathfrak{q}\}$ if $p \mid n$ and $\mathcal{P}_R^n = \mathcal{P}_R$ if $p \nmid n$.
(ii) If R_m is a unique factorisation domain for each $m \in D(n)$, then every prime in $\mathcal{P}_{R_m}$ is principal and so again one has $\mathcal{P}_R^n = \{\mathfrak{q} \in \mathcal{P}_R : p \notin \mathfrak{q}\}$ if $p \mid n$ and $\mathcal{P}_R^n = \mathcal{P}_R$ if $p \nmid n$.
(iii) If $\mathcal{O}_n \subseteq R$, then, for each $m \in D(n)$, the $\mathbb{Z}_p$-algebra R_m is a finite direct product of copies of R and so one has $\mathcal{P}_R^n = \{\mathfrak{q} \in \mathcal{P}_R^{\rm fg} : p \notin \mathfrak{q}\}$ if $p \mid n$ and $\mathcal{P}_R^n = \mathcal{P}_R^{\rm fg}$ if $p \nmid n$. In particular, in all cases one has $\mathcal{P}_R^n = \mathcal{P}_R^{\rm fg}$ for $n \in D(p-1)$.
(iv) Fix $\mathfrak{q} \in \mathcal{P}_R^{\rm fg}$ with $p \notin \mathfrak{q}$ and set $\kappa := R/\mathfrak{q}$. Fix a field E containing $Q(\kappa)$ and $\mathbb{Q}_p^c$ and, for $m \in D(n)$, set $F_m = Q(\kappa) \cap L_m \subseteq E$, write $\mathcal{O}'_m$ for the valuation ring of F_m and assume $\mathcal{O}'_n \subseteq \kappa$ (as occurs, for example, if either $F_n = \mathbb{Q}_p$ or κ is integrally closed in $Q(\kappa)$). Then $\mathcal{O}_m$ is a free $\mathcal{O}'_m$-module of rank $[L_m : F_m]$ so that $\kappa_m := \kappa \otimes_{\mathcal{O}'_m} \mathcal{O}_m$ is isomorphic to a subring of the field $Q(\kappa) \otimes_{F_m} L_m$ and hence (0) is its unique prime ideal lying over the zero ideal (0_κ) of κ. In particular, since the algebra $\kappa \otimes_{\mathbb{Z}_p} \mathcal{O}_m$ is a finite direct product of copies of κ_m, each prime ideal that lies over (0_κ) is principle and so each prime ideal of R_m that lies over $\mathfrak{q}$ is finitely generated. It follows that $\mathfrak{q} \in \mathcal{P}_R^n$.

From Proposition 2.6 we now obtain the following useful criterion.

Proposition 2.8. *Let M be an A-module for which the R-module $f^*(M)$ is finitely-presented, admissible and torsion. Then M is a finitely-presented, admissible torsion A-module if both $\mathcal{P}_R(f^*(M)) \subseteq \mathcal{P}_R^{|G|}$ and, in addition, $R_\mathfrak{q}$ is Noetherian for every $\mathfrak{q} \in \mathcal{P}_R(f^*(M))$.*

Proof. Under the stated assumptions, Proposition 2.6(iii) implies that the A-module M is finitely-presented and torsion and that $\mathcal{P}_A(M)$ is finite since $\mathcal{P}_R(f^*(M))$ is finite. Then, since $\mathcal{P}_A(M) \subseteq (f^*)^{-1}(\mathcal{P}_R(f^*(M)))$, Proposition 2.6(ii)(a) implies $\mathcal{P}_A(M) \subseteq \mathcal{P}_A^{\rm fg}$ if $\mathcal{P}_R(f^*(M)) \subseteq \mathcal{P}_R^{|G|}$. Finally we note that if $\mathfrak{q} \in \mathcal{P}_R(f^*(M))$ is such that $R_\mathfrak{q}$ is Noetherian, then it is a Noetherian valuation ring that is not a field (as $\mathrm{ht}(\mathfrak{q}) = 1$) and hence a discrete valuation ring. In this case, therefore, the $R_\mathfrak{q}$-order $R_\mathfrak{q}[G]$ is maximal if and only if $|G| \notin \mathfrak{q}$ (cf. [12, Prop. (27.1)]). The admissibility of M as an A-module now follows directly from Proposition 2.6(ii)(b) (and the first assertion of Remark 2.2(i)). $\square$

Remark 2.9. Fix a natural number n, let R be the completed p-adic group ring $\mathbb{Z}_p[[\mathbb{Z}_p^n]]$ and assume that p divides $|G|$. Then $A = R[G]$ is Noetherian (but neither integrally closed nor a domain), $Q(A)$ is semisimple and Proposition 2.8 combines with Example 2.7(i) to imply that a finitely generated torsion A-module M is admissible if $pR \notin \mathcal{P}_R(f^*(M))$. By the classical structure theory of Iwasawa modules (cf. [26, Prop. (5.1.7)(ii)]), this condition is satisfied if and only if the submodule $M[p^\infty]$ of M of elements of finite (p-power) order is pseudo-null. Hence, in this case, Theorem 2.3(ii)(b) provides the following 'equivariant' refinement of the structure theorem for Iwasawa modules: if M is a finitely generated torsion A-module for which $M[p^\infty]$ is pseudo-null, then $\mathcal{P}_A(M)$ is finite and M is pseudo-isomorphic, as an A-module, to a finite direct sum of modules of the form $A/\mathfrak{p}^{e(\mathfrak{p})}$, with $\mathfrak{p} \in \mathcal{P}_A(M)$ and $e(\mathfrak{p}) \in \mathbb{N}$.

2.2. **Generalised characteristic ideals.** In this section we assume $Q(A)$ is semisimple. Then, for any finitely-presented, admissible, torsion A-module M, the set $\mathcal{P}_A(M)$ is finite and, by Theorem 2.3(ii)(b), for each $\mathfrak{p}$ in $\mathcal{P}_A(M)$ there exists a finite set $\{e(\mathfrak{p})_i\}_{1\le i\le n(\mathfrak{p})}$ of natural numbers $e(\mathfrak{p})_i$ for which there exists a pseudo-isomorphism of A-modules

$$M \to \bigoplus\nolimits_{\mathfrak{p}\in\mathcal{P}_A(M)} \bigoplus\nolimits_{1\le i\le n(\mathfrak{p})} A/\mathfrak{p}^{e(\mathfrak{p})_i}. \tag{7}$$

In addition, the same result also implies the existence of a finite family of principal ideals $\{ L_\tau \}_{\tau\in\mathcal{T}}$ of A together with a pseudo-isomorphism of A-modules

$$M \to \bigoplus\nolimits_{\tau\in\mathcal{T}} A/L_\tau. \tag{8}$$

These pseudo-isomorphisms then naturally suggest the following definitions.

Definition 2.10. Assume $Q(A)$ is semisimple and let M be a finitely-presented, admissible, torsion A-module. Then the *lower* and *upper generalised characteristic ideals* of M (with respect to the pseudo-isomorphims (7) and (8)) are the ideals of A that are respectively obtained by setting

$$\mathrm{char}_A(M) := \prod\nolimits_{\mathfrak{p}\in\mathcal{P}_A(M)} \mathfrak{p}^{\sum_{1\le i\le n(\mathfrak{p})} e(\mathfrak{p})_i}.$$

and

$$\mathrm{Char}_A(M) := \prod\nolimits_{\tau\in\mathcal{T}} L_\tau.$$

The distinguishing features of these ideals are that $\mathrm{char}_A(M)$ is defined via an explicit product of primes in $\mathcal{P}_A$, whilst $\mathrm{Char}_A(M)$ is defined to be principal. In the next result, we discuss the relation between them (and, in particular, justify the 'lower' and 'upper' terminology) and their dependence on the respective choices of pseudo-isomorphism, and also show that they retain some of the key properties of the characteristic ideals in classical Iwasawa theory (and see also Remark 2.12 below).

In the sequel we write $\mathrm{Fit}^0_A(M)$ for the initial Fitting ideal of a finitely-presented A-module M. We also refer to M as 'quadratically-presented' if, for some natural number d, it lies in an exact sequence of A-modules of the form

$$A^d \xrightarrow{\theta} A^d \to M \to 0. \tag{9}$$

Proposition 2.11. *Assume $Q(A)$ is semisimple.*

(i) *If M is a finitely-presented, torsion A-module, then the following claims are valid.*
   (a) *If M is admissible, then $\mathrm{char}_A(M)$ is independent of the choice of pseudo-isomorphism (7) and one has $\mathrm{char}_A(M)_\mathfrak{p} = \mathrm{Char}_A(M)_\mathfrak{p}$ for all $\mathfrak{p}$ in $\mathcal{P}_A$.*

(b) *Assume $A = R[G]$, with R a $\mathbb{Z}_p$-algebra that is a Krull domain and G a finite abelian group. Then M is admissible if $\mathcal{P}_R(f^*(M)) \subseteq \mathcal{P}_R^{|G|}$. Assuming this to be the case, the following claims are also valid.*

(i) $\mathrm{Char}_A(M) = \bigcap_{\mathfrak{q}\in\mathcal{P}_R} f^*(\mathrm{char}_A(M))_{\mathfrak{q}}$. *In particular,* $\mathrm{Char}_A(M)$ *is independent of the choice of pseudo-isomorphism (8).*

(ii) $\mathrm{char}_A(M) \subseteq \mathrm{Char}_A(M)$, *with equality if and only if* $\mathrm{char}_A(M)$ *is principal. In addition, the quotient* $\mathrm{Char}_A(M)/\mathrm{char}_A(M)$ *is pseudo-null.*

(iii) *If M is quadratically-presented, then* $\mathrm{Char}_A(M) = \mathrm{Fit}_A^0(M)$.

(ii) *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence of finitely generated A-modules. Then the following claims are valid.*

(a) *If M_2 is a finitely-presented, admissible, torsion A-module, then M_3 is a finitely-presented, admissible, torsion A-module and* $\mathrm{char}_A(M_2) \subseteq \mathrm{char}_A(M_3)$.

(b) *If M_1 and M_3 are finitely-presented, admissible, torsion A-modules, then M_2 is a finitely-presented, admissible, torsion A-module and*

$$\mathrm{char}_A(M_2) = \mathrm{char}_A(M_1) \cdot \mathrm{char}_A(M_3).$$

Proof. To prove (i)(a) we fix $\mathfrak{p} \in \mathcal{P}_A(M)$ and note that, if M is admissible, then the ring $A_{\mathfrak{p}} = A'_{\mathfrak{p}'}$ that occurs in the proof of Theorem 2.3(ii)(a) is a discrete valuation ring. Writing $l_{\mathfrak{p}}(N)$ for the length of a finitely generated, torsion $A_{\mathfrak{p}}$-module N, one can then compute

$$\begin{aligned} e(\mathfrak{p}) := \sum_{1\le i\le n(\mathfrak{p})} e(\mathfrak{p})_i &= l_{\mathfrak{p}}\Big(\bigoplus_{1\le i\le n(\mathfrak{p})} A_{\mathfrak{p}}/(\mathfrak{p}A_{\mathfrak{p}})^{e(\mathfrak{p})_i}\Big) \\ &= l_{\mathfrak{p}}\Big(\bigoplus_{\mathfrak{a}\in\mathcal{P}_A(M)}\ \bigoplus_{1\le i\le n(\mathfrak{a})} (A/\mathfrak{a}^{e(\mathfrak{a})_i})_{\mathfrak{p}}\Big) = l_{\mathfrak{p}}(M_{\mathfrak{p}}), \end{aligned} \tag{10}$$

where the last equality follows from the pseudo-isomorphism (7). One therefore has

$$\mathrm{char}_A(M)_{\mathfrak{p}} = \mathfrak{p}^{e(\mathfrak{p})}A_{\mathfrak{p}} = (\mathfrak{p}A_{\mathfrak{p}})^{l_{\mathfrak{p}}(M_{\mathfrak{p}})}$$

which, in particular, implies the first assertion of (i)(a). In the same way, the pseudo-isomorphism (8) implies that each $A_{\mathfrak{p}}$-module $A_{\mathfrak{p}}/L_{\tau,\mathfrak{p}}$ is torsion and that

$$l_{\mathfrak{p}}(M_{\mathfrak{p}}) = \sum\nolimits_{\tau\in\mathcal{T}} l_{\mathfrak{p}}(A_{\mathfrak{p}}/L_{\tau,\mathfrak{p}}) = l_{\mathfrak{p}}\big(A_{\mathfrak{p}}/(\prod\nolimits_{\tau\in\mathcal{T}} L_\tau)_{\mathfrak{p}}\big) = l_{\mathfrak{p}}(A_{\mathfrak{p}}/\mathrm{Char}_A(M)_{\mathfrak{p}})$$

and hence $\mathrm{Char}_A(M)_{\mathfrak{p}} = (\mathfrak{p}A_{\mathfrak{p}})^{l_{\mathfrak{p}}(M_{\mathfrak{p}})} = \mathrm{char}_A(M)_{\mathfrak{p}}$. To complete the proof of (i)(a), it is now enough to note that if $\mathfrak{p} \in \mathcal{P}_A \setminus \mathcal{P}_A(M)$, then it is clear $\mathrm{char}_A(M)_{\mathfrak{p}} = A_{\mathfrak{p}}$ and also that the pseudo-isomorphism (8) implies $L_{\tau,\mathfrak{p}} = A_{\mathfrak{p}}$ for all $\tau \in \mathcal{T}$ and hence $\mathrm{Char}_A(M)_{\mathfrak{p}} = A_{\mathfrak{p}}$.

To prove (i)(b) we assume R is a Krull domain and $A = R[G]$. Then $\mathcal{P}_R(f^*(M))$ is finite and $f^*(M)$ is admissible if $\mathcal{P}_R(f^*(M)) \subseteq \mathcal{P}_R^{\mathrm{fg}}$ (cf. Remark 2.5(i)). By applying the argument of Proposition 2.6(ii) in this case, we deduce that M is admissible provided $\mathcal{P}_R(f^*(M)) \subseteq \mathcal{P}_R^{|G|}$ (as we assume henceforth).

Before proceeding, we next show that

$$f^*(\mathrm{char}_A(M))_{\mathfrak{q}} = f^*(\mathrm{Char}_A(M))_{\mathfrak{q}} \quad \text{for every } \mathfrak{q} \in \mathcal{P}_R. \tag{11}$$

For this, we first assume that $\mathfrak{q} \notin \mathcal{P}_R(f^*(M))$. Then one has $f^*(M)_{\mathfrak{q}} = (0)$ so that the pseudo-isomorphisms (7) and (8) imply $f^*(\mathfrak{p}^{e(\mathfrak{p})_i})_{\mathfrak{q}} = f^*(A)_{\mathfrak{q}} = f^*(L_\tau)_{\mathfrak{q}}$ for each $\mathfrak{p} \in \mathcal{P}_A(M)$, integer i with $1 \le i \le n(\mathfrak{p})$ and $\tau \in \mathcal{T}$. This in turn implies $f^*(\mathrm{char}_A(M))_\mathfrak{q} = f^*(A)_\mathfrak{q} = f^*(\mathrm{Char}_A(M))_\mathfrak{q}$. It is thus enough to verify (11) for $\mathfrak{q} \in \mathcal{P}_R(f^*(M))$. For such $\mathfrak{q}$ one has $|G| \notin \mathfrak{q}$ and so, in order to deduce (11) from the final assertion of (i)(a), it is enough to show that, for any such $\mathfrak{q}$ and any ideal X of A the module $f^*(X)_\mathfrak{q}$ is uniquely determined by $\{X_\mathfrak{p} : \mathfrak{p} \in (f^*)^{-1}(\mathfrak{q})\}$. To see this, we note the argument of Proposition 2.6(ii) implies $f^*(A)_\mathfrak{q} = \prod_{i\in I}\mathcal{O}'_i$, with each $\mathcal{O}'_i$ the integral closure in K_i of the discrete valuation ring $R_\mathfrak{q}$. There is also a natural bijection $j : (f^*)^{-1}(\mathfrak{q}) \to \bigcup_{i\in I}\Sigma(i)$, where $\Sigma(i)$ denotes the (finite) set of maximal ideals of $\mathcal{O}'_i$, such that $X_\mathfrak{p} = (f^*(X)_\mathfrak{q})_{j(\mathfrak{p})}$ for $\mathfrak{p} \in (f^*)^{-1}(\mathfrak{q})$. In addition, each ring $\mathcal{O}'_i$ is a principal ideal domain (as a Dedekind domain with only finitely many prime ideals) and equal to $\bigcap_{\mathfrak{B}\in\Sigma(i)}\mathcal{O}'_{i,\mathfrak{B}}$. In particular, $f^*(X)_\mathfrak{q} = \bigoplus_{i\in I} X(i)$, with each $X(i) := \mathcal{O}'_i \otimes_A X$ an ideal of $\mathcal{O}'_i$. In addition, $X(i) = (0)$ if and only if $X(i)_\mathfrak{B} = (0)$ for any $\mathfrak{B} \in \Sigma(i)$ and, if $X(i) \ne (0)$, then it is isomorphic to $\mathcal{O}'_i$ and hence equal to $\bigcap_{\mathfrak{B}\in\Sigma(i)} X(i)_\mathfrak{B}$. The claimed result is therefore true since $X(i)_\mathfrak{B} = X_{j^{-1}(\mathfrak{B})}$ for each $\mathfrak{B} \in \Sigma(i)$.

Next we observe that the claimed equality in (i)(b)(i) combines with the independence assertion in (i)(a) to directly imply the second assertion of (i)(b)(i). In addition, to prove both the equality in (i)(b)(i) and the first assertion of (i)(b)(ii) it is enough to show that

$$\mathrm{char}_A(M) \subseteq \bigcap\nolimits_{\mathfrak{q}\in\mathcal{P}_R} f^*(\mathrm{char}_A(M))_\mathfrak{q} = \bigcap\nolimits_{\mathfrak{q}\in\mathcal{P}_R} f^*(\mathrm{Char}_A(M))_\mathfrak{q} = \mathrm{Char}_A(M). \tag{12}$$

Here the inclusion is clear (since R is a domain) and the first equality follows from (11). Further, since R is a Krull domain, the second equality will follow if we can show that $\mathrm{Char}_A(M)$ is free as a (finitely generated) R-module. To prove this it is enough to show that the principal ideal $\mathrm{Char}_A(M)$ of A contains a non-zero divisor (of A). To do this, we note first that each $\mathfrak{p} \in \mathcal{P}_A(M)$ contains a non-zero divisor (as if $m \in M$ has non-zero image in $M_\mathfrak{p}$, then $\mathfrak{p}$ contains any non-zero divisor that annihilates m). This implies the existence of a non-zero divisor a in $\mathrm{char}_A(M)$. Then, for $\mathfrak{q} \in \mathcal{P}_R$, one has $a \in f^*(\mathrm{char}_A(M))_\mathfrak{q} = f^*(\mathrm{Char}_A(M))_\mathfrak{q}$ and so $ra = b$ for some $r \in R \setminus \mathfrak{q}$ and $b \in \mathrm{Char}_A(M)$. The element b is then a non-zero divisor of the sort required to complete the proof of (12).

In a similar way, if $\mathrm{char}_A(M)$ is a principal ideal, then it is a free R-module (as it contains a non-zero divisor) and so the first inclusion in (12) is an equality. This proves the second assertion of (i)(b)(ii) and the third assertion then follows directly from the final assertion of (i)(a). Lastly, to prove (i)(b)(iii) we note that, for $\mathfrak{p} \in \mathcal{P}_A(M)$, the presentation (9) gives rise to an exact sequence of $A_\mathfrak{p}$-modules

$$A_\mathfrak{p}^d \xrightarrow{\theta_\mathfrak{p}} A_\mathfrak{p}^d \to M_\mathfrak{p} \to 0. \tag{13}$$

Hence, since $M_\mathfrak{p}$ is a torsion module over the discrete valuation ring $A_\mathfrak{p}$, one has

$$A_\mathfrak{p} \cdot \det(\theta_\mathfrak{p}) = \mathfrak{p}_\mathfrak{p}^{l_\mathfrak{p}(\mathrm{coker}(\theta_\mathfrak{p}))} = \mathfrak{p}_\mathfrak{p}^{l_\mathfrak{p}(M_\mathfrak{p})} = \mathfrak{p}_\mathfrak{p}^{e(\mathfrak{p})} = \mathrm{Char}_A(M)_\mathfrak{p}. \tag{14}$$

Here the first equality is valid since $A_\mathfrak{p}$ is an elementary divisor ring, the second follows from (13), the third from (10) and the last from the definition of $\mathrm{char}_A(M)$ and the final assertion of (i)(a).

Now, since M is torsion, the exact sequence (9) implies $\det(\theta)$ is a unit of $Q(A)$ (and hence a non-zero divisor of A). This implies $f^*(A \cdot \det(\theta))$ is a (finitely generated) free

R-module and thereby implies the equality in (i)(b)(iii) via the computation

$$\mathrm{Fit}_A^0(M) = A \cdot \det(\theta) = \bigcap_{\mathfrak{q}\in\mathcal{P}_R} f^*(A\cdot\det(\theta))_{\mathfrak{q}} = \bigcap_{\mathfrak{q}\in\mathcal{P}_R} f^*(\mathrm{Char}_A(M))_{\mathfrak{q}} = \mathrm{Char}_A(M).$$

Here the first equality follows directly from the definition of initial Fitting ideal (and the resolution (9)), the second from the fact R is a Krull domain and the last from (12). In addition, since $(A\cdot\det(\theta))_{\mathfrak{p}} = A_{\mathfrak{p}}\cdot\det(\theta_{\mathfrak{p}})$ for all $\mathfrak{p}\in\mathcal{P}_A$, the third equality is true since the equalities (14) imply that $f^*(A\cdot\det(\theta))_{\mathfrak{q}} = f^*(\mathrm{Char}_A(M))_{\mathfrak{q}}$ for all $\mathfrak{q}\in\mathcal{P}_R$ (in just the same way that the final assertion of (i)(a) implies (11)). This completes the proof of (i)(b).

Turning to (ii), we note that the assertions regarding modules being torsion and finitely-presented follow directly from the given exact sequence (and, in the latter case, the general result of [21, Th. 2.1.2]). In addition, for each prime ideal $\mathfrak{p}$ of A, the given sequence induces a short exact sequence of $A_{\mathfrak{p}}$-modules

$$0 \to M_{1,\mathfrak{p}} \to M_{2,\mathfrak{p}} \to M_{3,\mathfrak{p}} \to 0.$$

Assuming M_2 (or equivalently, both M_1 and M_3) to be torsion, these sequences imply an equality $\mathcal{P}(M_2) = \mathcal{P}(M_1)\cup\mathcal{P}(M_3)$ that combines with Remark 2.2 to imply both of the assertions regarding admissibility, and also combines with the observation made in the proof of (i)(a) to imply the stated inclusion, respectively equality, of characteristic ideals. □

Remark 2.12. Fix natural numbers m and n and write R for the completed group ring $\mathbb{Z}_p[\zeta_m][[\mathbb{Z}_p^n]]$. Then R is both Noetherian and admissible in the sense of Definition 2.4 (for example, by Remark 2.5(ii)) and, in addition, every prime in $\mathcal{P}_R$ is principal. In this case, therefore, the argument of Proposition 2.11(i)(b) has two concrete consequences. Firstly, if $p\nmid |G|$, then the ring $R[G]$ is admissible (by Example 2.7(i)). Secondly, for every finitely generated (and hence finitely presented by Noetherianity), torsion R-module M, the ideals $\mathrm{char}_R(M)$ and $\mathrm{Char}_R(M)$ are equal and are easily seen to coincide with the classical characteristic ideal of M as an R-module.

2.3. **Inverse limit rings.** In this section we assume to be given an inverse system of rings

$$(A_n,\ \phi_n\colon A_n\to A_{n-1})_{n\in\mathbb{N}}$$

in which every homomorphism ϕ_n is surjective. We study the associated inverse limit ring

$$A := \varprojlim_n A_n.$$

For every n we write $\phi_{\langle n\rangle}: A\to A_n$ for the induced (surjective) projection map, so that $\phi_n\circ\phi_{\langle n\rangle} = \phi_{\langle n-1\rangle}$ for all n, and we use the decreasing separated filtration

$$I_\bullet := (I_n)_{n\in\mathbb{N}}$$

of A that is obtained by setting $I_n := \ker(\phi_{\langle n\rangle})$ for every n. For an A-module M and non-negative integer n, we then define an A_n-module by setting

$$M_{(n)} := M/(I_n\cdot M) \cong (A/I_n)\otimes_A M \cong A_n\otimes_A M.$$

We also use similar notation for morphisms, so that $\theta_{(n)}: M_{(n)}\to N_{(n)}$ denotes the morphism $\mathrm{id}_{A_n}\otimes_A\theta$ induced by a given morphism of A-modules $\theta: M\to N$.

We say M is '$I_\bullet$-complete' if the natural map

$$\mu_M : M \to \varprojlim_n M_{(n)}$$

is bijective, where the inverse limit is taken with respect to the maps $\phi_{M,n} : M_{(n)} \to M_{(n-1)}$ induced by ϕ_n.

2.3.1. *The general case.* The following result records some useful general facts about the notion of $I_\bullet$-completeness. In this result we refer to the linear topology on A induced by the subgroups $\{I_n\}_n$ as the '$I_\bullet$-topology'.

Lemma 2.13. *The following claims are valid for every A-module M.*

(i) *If M is finitely generated, then μ_M is surjective but need not be injective.*
(ii) *M is $I_\bullet$-complete if it is a finitely generated submodule of an $I_\bullet$-complete module. In particular, every finitely generated ideal of A is $I_\bullet$-complete.*
(iii) *Assume M is $I_\bullet$-complete and that there exists a natural number t for which both the I_t-adic topology on A is finer than the $I_\bullet$-topology and the A_t-module $M_{(t)}$ is finitely generated. Then M is generated as an A-module by any finite subset that projects to give a set of generators of $M_{(t)}$.*

Proof. To prove (i) we fix a natural number d for which there exists an exact sequence of A-modules of the form

$$0 \to K \xrightarrow{\subseteq} A^d \xrightarrow{\varphi} M \to 0. \tag{15}$$

For each n, we set $K'_n := \ker(\varphi_{(n)})$ and use the exact commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & K'_n \arrow[r, "\subseteq"] \arrow[d, "\alpha_n"] & A^d_{(n)} \arrow[r, "\varphi_{(n)}"] \arrow[d, "(\phi_n)^d"] & M_{(n)} \arrow[r] \arrow[d, "\phi_{M,n}"] & 0 \\ 0 \arrow[r] & K'_{n-1} \arrow[r, "\subseteq"] & A^d_{(n-1)} \arrow[r, "\varphi_{(n-1)}"] & M_{(n-1)} \arrow[r] & 0. \end{tikzcd}$$

Write $I_{[n]}$ for the image of I_{n-1} in A_n. Then $\ker((\phi_n)^d) = I^d_{[n]}$ and $\ker(\phi_{M,n}) = I_{[n]} \cdot M_{(n)}$. Thus, since each map $(\phi_n)^d$ is surjective, the Snake Lemma applies to the above diagram to imply that each map α_n is also surjective. By passing to the limit over n of these diagrams we thus obtain the bottom row of the exact commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[d] & A^d \arrow[r, "\varphi"] \arrow[d, "(\mu_A)^d"] & M \arrow[r] \arrow[d, "\mu_M"] & 0 \\ 0 \arrow[r] & \varprojlim_n K'_n \arrow[r] & (\varprojlim_n A_{(n)})^d \arrow[r] & \varprojlim_n M_{(n)} \arrow[r] & 0. \end{tikzcd} \tag{16}$$

In addition, for each n the (surjective) map $\phi_{\langle n\rangle}$ induces an isomorphism $A_{(n)} \cong A_n$ so that the map $(\mu_A)^d$ is bijective (and hence A^d is $I_\bullet$-complete). From the above diagram, one can therefore deduce that μ_M is surjective.

To give an example in which μ_M is not injective we take A_n to be the power series ring $\mathbb{Z}_p[[X_1, \dots, X_n]]$ over $\mathbb{Z}_p$ in n commuting indeterminates X_i and ϕ_n to be the projection map $A_n \to A_{n-1}$ induced by sending X_n to 0. In this case A identifies with one version (see

[10]) of the power series ring over $\mathbb{Z}_p$ in a countable number of commuting indeterminates $\{X_i\}_{i\in\mathbb{N}}$. We then define K to be the (proper) ideal of A that is generated by the set $\{pX_1\}\cup\{X_n - pX_{n+1}\}_{n\in\mathbb{N}}$ and take M to be the quotient A/K. In this case, one computes that the modules $M_{(n)}$ lie in commutative diagrams of the form

$$\begin{array}{ccccc} M_{(n)} & \xrightarrow{\sim} & A_n/\phi_{\langle n\rangle}(K) & \xrightarrow{\sim} & \mathbb{Z}_p \\ {\scriptstyle\phi_{M,n}}\downarrow & & \downarrow & & \| \\ M_{(n-1)} & \xrightarrow{\sim} & A_{n-1}/\phi_{\langle n\rangle}(K) & \xrightarrow{\sim} & \mathbb{Z}_p. \end{array}$$

Here the first horizontal map in the upper row is the isomorphism induced by $\phi_{\langle n\rangle}$ and the second is the isomorphism induced by the fact $\phi_{\langle n\rangle}(K)$ is equal to the ideal of A_n generated by $(X_1,\dots,X_n)$, with similar descriptions for the maps in the lower row, and the unlabelled vertical map is induced by ϕ_n. These diagrams imply that $\ker(\mu_M) = \ker(\epsilon)/K$, with ϵ the projection $\epsilon : A \to \mathbb{Z}_p$ that sends each X_i to 0, and hence that μ_M is not injective since $X_1 \in \ker(\epsilon)\setminus K$.

To prove the first assertion of (ii) we fix an injective map $\theta : M \to N$ in which N is $I_\bullet$-complete. It is then enough to note that μ_M is injective as a consequence of the diagram

$$\begin{array}{ccc} M & \xhookrightarrow{\theta} & N \\ \downarrow{\scriptstyle\mu_M} & & \downarrow{\scriptstyle\mu_N} \\ \varprojlim_n M_{(n)} & \xrightarrow{(\theta_{(n)})_n} & \varprojlim_n N_{(n)} \end{array}$$

and the fact that μ_N is injective. The second assertion of (ii) is then an immediate consequence of the fact A is $I_\bullet$-complete (as shown above).

To prove (iii) we mimic the argument of [24, Th. 8.4]. To do this we fix a finite set of elements $\{m_\sigma\}_{\sigma\in\Sigma}$ of M with $M = (\sum_{\sigma\in\Sigma} Am_\sigma) + I_t\cdot M$. Then $M = (\sum_{\sigma\in\Sigma} Am_\sigma) + I_t^n\cdot M$ for every n and so, since for each $n\in\mathbb{N}$ there exists (by assumption) $n_1\in\mathbb{N}$ with $(I_t)^{n_1}\subseteq I_n$, one therefore also has

$$M = (\sum\nolimits_{\sigma\in\Sigma} Am_\sigma) + I_n\cdot M \quad \text{for every } n. \tag{17}$$

We now fix $m\in M$ and set $m_0 := m$ and $I_0 := A$. Then, for each $n\in\mathbb{N}$, we inductively choose $\{a_{\sigma,n}\}_{\sigma\in\Sigma}\subseteq I_{n-1}$ and $m_n\in I_{n-1}I_n\cdot M \subset I_n\cdot M$ with $m_{n-1} = (\sum_{\sigma\in\Sigma} a_{\sigma.n}m_\sigma) + m_n$. That such elements can be chosen for $n=1$ is a direct consequence of (17) with $n=1$. Then, if one assumes their existence for $n=n_0$, their existence for n_0+1 is a consequence of the equality obtained after multiplying (17) with $n=n_0+1$ by I_{n_0}. Now, since A is $I_\bullet$-complete, for each $\sigma\in\Sigma$, there exists a unique element $a_\sigma\in A$ such that $a_\sigma - \sum_{i=1}^{i=n} a_{\sigma,i}\in I_n$ for all n. Then one checks that

$$m - (\sum\nolimits_{\sigma\in\Sigma} a_\sigma m_i) \in \bigcap\nolimits_{n\in\mathbb{N}} (I_n\cdot M) = (0)$$

where the last equality is valid since M is $I_\bullet$-complete. This shows that M is generated over A by $\{m_\sigma\}_{\sigma\in\Sigma}$, as required. □

2.3.2. *The compact case.* In the sequel we shall say that the inverse limit A is 'compact' if each ring A_n is endowed with a compact Hausdorff topology with respect to which the transition maps ϕ_n are continuous. In this case we endow A with the corresponding inverse limit topology, so that A is compact and Hausdorff and, for every n, the ideal I_n is closed and the projection map $\phi_{\langle n\rangle}$ is continuous.

In particular, since A is compact, the inverse limit functor is exact on the category of finitely generated A-modules and this fact allows us to prove a finer version of Lemma 2.13.

Before stating the result, we note that if an A-module N is pseudo-null, then the associated A_n-module $N_{(n)}$ need not even be torsion. Such issues mean that, in general, one cannot hope to compute the characteristic ideal of a finitely-presented torsion A-module M directly in terms of the associated A_n-modules $M_{(n)}$.

Despite this difficulty, claim (iii) of the following result shows that such a reduction is possible for a natural family of compact rings A, at least after possibly replacing M by a pseudo-isomorphic module. (In Proposition 3.4 below we will also prove a more concrete version of this result for certain power series rings.)

Proposition 2.14. *Assume that A is compact. Then the following claims are valid for any finitely-presented A-module M.*

(i) *M is $I_\bullet$-complete.*

(ii) *If M is an admissible, torsion module, then*

$$\mathrm{char}_A(M) = \varprojlim_n \phi_{\langle n\rangle}(\mathrm{char}_A(M)) \quad \textit{and} \quad \mathrm{Char}_A(M) = \varprojlim_n \phi_{\langle n\rangle}(\mathrm{Char}_A(M)),$$

where the limits are taken with respect to the maps ϕ_n.

(iii) *Assume A and A_n for each n are $\mathbb{Z}_p$-algebras and unique factorisation domains. Let M be a finitely-presented, torsion A-module. Then M is pseudo-isomorphic to an A-module $\widetilde{M}$ with the following properties: $\widetilde{M}$ is finitely-presented, torsion and $I_\bullet$-complete; there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, the A_n-module $\widetilde{M}_{(n)}$ is finitely-presented and torsion; one has*

$$\mathrm{Char}_A(M) = \mathrm{char}_A(M) = \varprojlim_{n\geq n_0} \mathrm{char}_{A_n}(\widetilde{M}_{(n)}),$$

where the limit is taken with respect to the maps ϕ_n.

Proof. To prove (i) we fix an exact sequence of A-modules of the form (15). Then the A-module K is, by assumption, finitely generated and thus, by Lemma 2.13(ii), $I_\bullet$-complete. Hence, by passing to the limit over n of the induced exact sequences of (compact) A_n-modules $K_{(n)} \to A_n^d \to M_{(n)} \to 0$ one obtains an exact sequence of A-modules

$$0 \to K \xrightarrow{\subseteq} A^d \to \varprojlim_n M_{(n)} \to 0.$$

Comparing this to (15) one deduces the map μ_M is bijective, as required to prove (i).

In the rest of the argument we assume M is torsion. Then, since $\mathrm{char}_A(M)$ and $\mathrm{Char}_A(M)$ are both finitely generated ideals of A (cf. condition (P_2) in Definition 2.1), to prove (ii) it is enough to show that any finitely generated ideal N of A is equal to $\varprojlim_n \phi_{\langle n\rangle}(N)$, where the limit is taken with respect to the maps ϕ_n. To see this, we note that the above argument (with $M = A/N$, $d = 1$ and $K = N$) implies that the map $\mu_{A/N}$ is bijective. The stated equality then follows from the corresponding exact commutative diagram (16) and the fact that, in this case, one has $K'_n = \phi_{\langle n\rangle}(N)$ for every n.

To prove (iii) we note that if B is equal to either A or A_n for any n, then the given assumptions imply it is admissible (cf. Example 2.5(ii)) and also that every ideal in $\mathcal{P}_B$ is principal so that, for any finitely-presented torsion B-module N, one has $\mathrm{Char}_B(N) = \mathrm{char}_B(N)$ (by Proposition 2.11(i)(b)(ii) with $R = B$ and G trivial). In addition, by Theorem 2.3(ii)(b), any finitely-presented torsion A-module M is pseudo-isomorphic to a finite direct sum $\widetilde{M} := \bigoplus_{\tau\in\mathcal{T}} A/L_\tau$, where, for each τ, $L_\tau = A\cdot a_\tau$ with $a_\tau \in A\setminus\{0\}$. In particular, $\widetilde{M}$ is finitely-presented and torsion and thus also $I_\bullet$-complete by (i). Further, for every n there is a natural isomorphism

$$\widetilde{M}_{(n)} \cong \bigoplus\nolimits_{\tau\in\mathcal{T}} \big(A/L_\tau\big)_{(n)} \cong \bigoplus\nolimits_{\tau\in\mathcal{T}} A_n/\phi_{\langle n\rangle}(L_\tau) = \bigoplus\nolimits_{\tau\in\mathcal{T}} A_n/(A_n\cdot\phi_{\langle n\rangle}(a_\tau)). \tag{18}$$

In particular, if n_0 is the smallest integer for which $\phi_{\langle n\rangle}(a_\tau) \neq 0$ for all $\tau\in\mathcal{T}$, then for every $n \geq n_0$ the A_n-module $\widetilde{M}_{(n)}$ is finitely-presented and torsion. It is then enough to note that

$$\mathrm{Char}_A(M) = \prod\nolimits_{\tau\in\mathcal{T}} L_\tau = \varprojlim_n \prod\nolimits_{\tau\in\mathcal{T}} \phi_{\langle n\rangle}(L_\tau) = \varprojlim_n \mathrm{char}_{A_n}(\widetilde{M}_{(n)}).$$

Here the first equality follows directly from our definition of upper generalised characteristic ideal, the second from (ii) and the third is valid since, for each n, the isomorphism (18) combines with Proposition 2.11(i)(b) to imply that

$$\mathrm{char}_{A_n}(\widetilde{M}_{(n)}) = \mathrm{Char}_{A_n}(\widetilde{M}_{(n)}) = \prod\nolimits_{\tau\in\mathcal{T}} \phi_{\langle n\rangle}(L_\tau).$$

□

3. Weil-étale cohomology for curves over finite fields

In this section we describe an application of the above results to the Iwasawa theory of curves over finite fields.

For this, we write $\mathcal{U}(G)$ for the set of open subgroups of a profinite group G.

3.1. Galois groups and power series rings. The Iwasawa algebra of $\mathbb{Z}_p^{\mathbb{N}}$ over a commutative $\mathbb{Z}_p$-algebra $\mathcal{O}$ is the completed group ring

$$\mathcal{O}[[\mathbb{Z}_p^{\mathbb{N}}]] := \varprojlim_{U\in\mathcal{U}(\mathbb{Z}_p^{\mathbb{N}})} \mathcal{O}[\mathbb{Z}_p^{\mathbb{N}}/U],$$

where the limit is taken respect to the natural projection maps.

In particular, after fixing a $\mathbb{Z}_p$-basis $\{\gamma_i\}_{i\in\mathbb{N}}$ of $\mathbb{Z}_p^{\mathbb{N}}$, the association $X_i \mapsto \gamma_i - 1$ induces a (non-canonical) isomorphism of rings between $\mathcal{O}[[\mathbb{Z}_p^{\mathbb{N}}]]$ and the power series ring

$$\mathcal{R}_{\mathcal{O}} := \varprojlim_n \mathcal{R}_{n,\mathcal{O}} \quad \text{with} \quad \mathcal{R}_{n,\mathcal{O}} := \mathcal{O}[[X_1,\ldots,X_n]]$$

in commuting indeterminants $\{X_i\}_{i\in\mathbb{N}}$. Here the inverse limit is taken with respect to the (surjective) $\mathbb{Z}_p$-linear ring homomorphisms

$$\rho_{n,\mathcal{O}} : \mathcal{R}_{n,\mathcal{O}} \twoheadrightarrow \mathcal{R}_{n-1,\mathcal{O}}$$

that send X_i to X_i if $1 \le i < n$ and to 0 if $i = n$. For each n we also use the maps

$$\iota_{n,\mathcal{O}} : \mathcal{R}_{n,\mathcal{O}} \hookrightarrow \mathcal{R}_{\mathcal{O}} \quad \text{and} \quad \rho_{\langle n\rangle,\mathcal{O}} : R_{\mathcal{O}} \twoheadrightarrow R_{n,\mathcal{O}},$$

that are respectively the natural inclusion and the (surjective) $\mathcal{O}$-linear ring homomorphism that sends X_i to X_i if $1 \le i \le n$ and to 0 if $i > n$ (so that the pair $(\iota_{n,\mathcal{O}}, \rho_{\langle n\rangle,\mathcal{O}})$ is a retract of rings and, for each $n > 1$, one has $\rho_{n,\mathcal{O}} \circ \rho_{\langle n\rangle,\mathcal{O}} = \rho_{\langle n-1\rangle,\mathcal{O}}$).

In the case $\mathcal{O} = \mathbb{Z}_p$, we abbreviate $\mathcal{R}_{\mathcal{O}}, \mathcal{R}_{n,\mathcal{O}}, \rho_{n,\mathcal{O}}, \rho_{\langle n\rangle,\mathcal{O}}$ and $\iota_{n,\mathcal{O}}$ to $\mathcal{R}, \mathcal{R}_n, \rho_n, \rho_{\langle n\rangle}$ and ι_n respectively. We then also fix a finite abelian group G and consider the group rings

$$\mathcal{A} := \mathcal{R}[G] \quad \text{and} \quad \mathcal{A}_n = \mathcal{R}_n[G],$$

together with the maps $\mathcal{A}_n \to \mathcal{A}_{n-1}, \mathcal{A}_n \to \mathcal{A}$ and $\mathcal{A} \to \mathcal{A}_n$ that are respectively induced by ρ_n, ι_n and $\rho_{\langle n\rangle}$ (and which we continue to denote by the same notation).

We then define a separated decreasing filtration $\mathcal{I}_\bullet = (\mathcal{I}_n)_n$ of $\mathcal{A}$ by setting

$$\mathcal{I}_n := \ker(\rho_{\langle n\rangle})$$

for each n, and we note that $\mathcal{A}$ is $\mathcal{I}_\bullet$-complete.

Now, since the submodule of $\mathcal{I}_n$ that is generated by $\{X_i\}_{i>n}$ is not finitely generated, the ring $\mathcal{A}$ is not Noetherian (cf. Remark 3.3 below) and its module theory is complicated. For instance, the example discussed in the proof of Lemma 2.13(i) shows that cyclic $\mathcal{A}$-modules need not be $\mathcal{I}_\bullet$-complete (or even pro-finite) and also, taking account of a result of Fujiwara et al [18, Th. 4.2.2], that $\mathcal{A}$ does not have the weak Artin-Rees property relative to p. Nevertheless, claims (i) and (ii) of the following result ensure that the theory developed in §2 can be applied in this setting.

We recall (from §2.1.2) that, for each natural number m, $\mathcal{O}_m$ denotes $\mathbb{Z}_p[\zeta_m] \subset \mathbb{Q}_p^c$.

Lemma 3.1. *For every n the following claims are valid.*

(i) *For all natural numbers m, the rings $\mathcal{R}_{\mathcal{O}_m}$ and $\mathcal{R}_{n,\mathcal{O}_m}$ are p-adically complete unique factorisation domains, and hence admissible (in the sense of Definition 2.4).*

(ii) *The ring $\mathcal{A}$ is p-adically complete and compact (in the sense of §2.3.2) and both rings $Q(\mathcal{A})$ and $Q(\mathcal{A}_n)$ are semisimple (as algebras over $Q(\mathcal{R})$ and $Q(\mathcal{R}_n)$ respectively). In addition, an $\mathcal{A}$-module M is finitely-presented, torsion and admissible if it is finitely-presented and torsion as an $\mathcal{R}$-module and, in addition, no height one prime of $\mathcal{R}$ that lies in the support of M contains $|G|$. In particular, if $p \nmid |G|$, then the ring $\mathcal{A}$, and also the ring $\mathcal{A}_n$ for each n, is admissible.*

(iii) *If $\mathfrak{p}$ is a prime ideal of $\mathcal{A}_n$, then $\iota_n(\mathfrak{p})\mathcal{A}$ is a prime ideal of $\mathcal{A}$.*

Proof. Since $\mathcal{O}_m$ is a regular local domain, the first assertion of (i) is classical in the case of $\mathcal{R}_{n,\mathcal{O}_m}$. This result then implies that the ring $\mathcal{R}_{\mathcal{O}_m}$ satisfies the condition $(*)$ of Nishimura [27, Intro.] and hence that it is a unique factorisation domain by [27, Th. 1]. The second assertion of (i) then follows directly from Remark 2.5(ii).

To prove (ii) we note that, for each subgroup U in $\mathcal{U}(\mathbb{Z}_p^{\mathbb{N}})$ the ring $\mathbb{Z}_p[(\mathbb{Z}_p^{\mathbb{N}}/U) \times G]$ is finitely generated over $\mathbb{Z}_p$ and hence compact with respect to the canonical p-adic topology. The (inverse limit) ring $\mathbb{Z}_p[[\mathbb{Z}_p^{\mathbb{N}} \times G]]$ is therefore compact with respect to the induced inverse limit topology. This induces a compact topology on $\mathcal{A}$ that is independent of the choice of $\mathbb{Z}_p$-basis $\{\gamma_i\}_{i\in\mathbb{N}}$ of $\mathbb{Z}_p^{\mathbb{N}}$ and such that each ideal $\mathcal{I}_n$ is closed. This proves the first assertion of (ii). In addition, as $\mathcal{R}$ and $\mathcal{R}_n$ are both domains of characteristic zero, and G is finite, the algebras $Q(\mathcal{A})$ and $Q(\mathcal{A}_n)$ are respectively equal to $Q(\mathcal{R})[G]$ and $Q(\mathcal{R}_n)[G]$ and so are semisimple (see the discussion at the beginning of §2.1.2).

Next we note that (i) combines with Proposition 2.8 (with R and A replaced by $\mathcal{R}$ and $\mathcal{A}$) to imply an $\mathcal{A}$-module M that is finitely-presented and torsion as an $\mathcal{R}$-module is finitely-presented, torsion and admissible as an $\mathcal{A}$-module provided that both $\mathcal{P}_{\mathcal{R}}(M) \subseteq \mathcal{P}_{\mathcal{R}}^{|G|}$ and $R_{\mathfrak{q}}$ is Noetherian for every $\mathfrak{q} \in \mathcal{P}_{\mathcal{R}}(M)$. In addition, since for each divisor m of n, the ring $\mathcal{O}_m \otimes_{\mathbb{Z}_p} \mathcal{R} = \mathcal{R}_{\mathcal{O}_m}$ is a unique factorisation domain, one has $\mathcal{P}_{\mathcal{R}}^{|G|} = \{\mathfrak{q} \in \mathcal{P}_{\mathcal{R}} : |G| \notin \mathfrak{q}\}$ (cf. Example 2.7(ii)) and the localisation of $\mathcal{R}$ at each prime in $\mathcal{P}_{\mathcal{R}}$ is a principal ideal domain, and hence Noetherian. This proves the second sentence of (ii). Given this fact, it is clear that if $p \nmid |G|$ then $\mathcal{A}$ is admissible as no prime in $\mathcal{P}_{\mathcal{R}}$ can contain $|G|$. Finally, we recall that the admissibility of each ring $\mathcal{A}_n$ in this case was already observed in Remark 2.12

To prove (iii) we note $\mathfrak{p}$ is a (finitely generated) ideal of the (Noetherian) ring $\mathcal{A}_n$, and hence that $\mathfrak{P} := \iota_n(\mathfrak{p})\mathcal{A}$ is a finitely generated ideal of $\mathcal{A}$. Proposition 2.14(i) therefore implies that the map $\mu_{\mathcal{A}/\mathfrak{P}}$ is bijective. Since, for $m > n$, the image of the natural map $\mathfrak{P}_{(m)} \to \mathcal{A}_{(m)} = \mathcal{A}_m$ is $\rho_{\langle m\rangle}(\mathfrak{P}) = \mathfrak{p}[[X_{n+1}, \dots, X_m]]$, these observations combine to give a composite ring isomorphism

$$\mathcal{A}/\mathfrak{P} \xrightarrow{\mu_{\mathcal{A}/\mathfrak{P}}} \varprojlim_{m>n} \left(\mathcal{A}/\mathfrak{P}\right)_{(m)} \cong \varprojlim_{m>n} \mathcal{A}_m/\rho_{\langle m\rangle}(\mathfrak{P}) \cong \varprojlim_{m>n} (\mathcal{A}_n/\mathfrak{p})[[X_{n+1}, \cdots, X_m]].$$

Hence, since each ring $(\mathcal{A}_n/\mathfrak{p})[[X_{n+1}, \cdots, X_m]]$ is a domain, the limit is a domain and so $\mathfrak{P}$ is a prime ideal of $\mathcal{A}$. □

Remark 3.2. Every non-zero prime ideal of $\mathcal{R}$ that is principal has height one (since if a generating element x does not belong to any prime in $\mathcal{P}_{\mathcal{R}}$, then x^{-1} belongs to $\mathcal{R}_{\mathfrak{q}}$ for all $\mathfrak{q}$ in $\mathcal{P}_{\mathcal{R}}$ and hence to $\mathcal{R} = \bigcap_{\mathfrak{q}\in\mathcal{P}_{\mathcal{R}}} \mathcal{R}_{\mathfrak{q}}$). Lemma 3.1(iii) (with G trivial) therefore implies that $\iota_n(\mathfrak{p})\mathcal{R}$ belongs to $\mathcal{P}_{\mathcal{R}}$ if $\mathfrak{p}$ belongs to $\mathcal{P}_{\mathcal{R}_n}$. This observation is a special case of a result of Gilmer [19, Th. 3.2] and is also related to the second part of [2, Prop. 2.3].

Remark 3.3. Since $\mathcal{R}$ is a unique factorisation domain, it is a finite conductor ring in the sense of Glaz [20] (so that every ideal with at most two generators is finitely-presented). However, as far as we are aware, it is still not known whether $\mathcal{R}$ is a coherent ring.

The following result proves a more concrete version of Proposition 2.14(iii) in this case. In particular, it shows that, for a natural class of torsion $\mathcal{A}$-modules, the notion of lower generalised characteristic ideal coincides with the 'pro-characteristic ideal' defined by Bandini et al in [2].

Proposition 3.4. *Assume $|G|$ is prime to p. Then the following claims are valid for any quadratically-presented, torsion $\mathcal{A}$-module M.*

- (i) *For any natural number n for which the $\mathcal{A}_n$-module $M_{(n)}$ is torsion, the $\mathcal{A}_n$-module $(M_{(n+1)})^{X_{n+1}=0}$ is pseudo-null.*
- (ii) *The $\mathcal{A}$-module M identifies with $\varprojlim_n M_{(n)}$ and its pro-characteristic ideal (in the sense of* [2, Def. 1.3]*) is equal to* $\mathrm{char}_{\mathcal{A}}(M)$.

Proof. Since $p \nmid |G|$, there exists a finite set $\{m_i\}_{i\in I}$ of natural numbers and corresponding direct product decompositions $\mathcal{A} = \prod_{i\in I} \mathcal{R}_{\mathcal{O}_{m_i}}$ and $\mathcal{A}_n = \prod_{i\in I} \mathcal{R}_{n,\mathcal{O}_{m_i}}$ (for each n) that are compatible with all transition maps. Hence, in this argument we can, and will, henceforth assume that $\mathcal{A}$ and $\mathcal{A}_n$ respectively represent $\mathcal{R}_{\mathcal{O}_m}$ and $\mathcal{R}_{n,\mathcal{O}_m}$ for some natural number m.

To prove (i) we note $\mathcal{A}_{n+1}$ is Noetherian. Hence, assuming $M_{(n)}$ to be a torsion $\mathcal{A}_n$-module, the equality $(M_{(n+1)})_{(n)} = M_{(n)}$ combines with Nakayama's Lemma to imply $(M_{(n+1)})_{\mathfrak{p}} = (0)$ with $\mathfrak{p} = (X_{n+1}) \in \mathrm{Spec}(\mathcal{A}_{n+1})$ and so $M_{(n+1)}$ is a torsion $\mathcal{A}_{n+1}$-module. In particular, since $M_{(n+1)}$ and $M_{(n)}$ are both quadratically-presented (over $\mathcal{A}_{n+1}$ and $\mathcal{A}_n$ respectively), there are equalities of $\mathcal{A}_n$-ideals

$$\begin{aligned}\mathrm{char}_{\mathcal{A}_n}\big((M_{(n+1)})^{X_{n+1}=0}\big)\cdot\rho_{n+1}\big(\mathrm{char}_{\mathcal{A}_{n+1}}(M_{(n+1)})\big) &= \mathrm{char}_{\mathcal{A}_n}(M_{(n)}) \\ &= \mathrm{Fit}^0_{\mathcal{A}_n}(M_{(n)}) \\ &= \rho_{n+1}\big(\mathrm{Fit}^0_{\mathcal{A}_{n+1}}(M_{(n+1)})\big) \\ &= \rho_{n+1}\big(\mathrm{char}_{\mathcal{A}_{n+1}}(M_{(n+1)})\big).\end{aligned} \tag{19}$$

Here the second and last equalities follow from Proposition 2.11(i)(b) (with G trivial and R taken to be respectively $\mathcal{A}_n$ and $\mathcal{A}_{n+1}$), the first equality follows from Remark 2.12 and the general result of [2, Prop. 2.10] (see also [28, Lem. 4]) and the third from a standard property of Fitting ideals under scalar extension.

Next we note that, as $M_{(n)}$ is a quadratically-presented, torsion $\mathcal{A}_n$-module, the ideal $\mathrm{Fit}^0_{\mathcal{A}_n}\big(M_{(n)}\big)$, and hence (by (19)) also $\rho_{n+1}\big(\mathrm{char}_{\mathcal{A}_{n+1}}(M_{(n+1)})\big)$, is principal and generated by a non-zero divisor. The equalities (19) therefore imply $\mathrm{char}_{\mathcal{A}_n}\big((M_{(n+1)})^{X_{n+1}=0}\big) = \mathcal{A}_n$, and hence that $(M_{(n+1)})^{X_{n+1}=0}$ is a pseudo-null $\mathcal{A}_n$-module, as required to prove (i).

In a similar way, Proposition 2.11(i)(b) implies for every n that

$$\mathrm{char}_{\mathcal{A}_n}(M_{(n)}) = \mathrm{Fit}^0_{\mathcal{A}_n}(M_{(n)}) = \rho_{\langle n\rangle}\big(\mathrm{Fit}^0_{\mathcal{A}}(M)\big) = \rho_{\langle n\rangle}\big(\mathrm{char}_{\mathcal{A}}(M)\big).$$

Taking account of Proposition 2.14(ii) (and Lemma 3.1(ii)), these equalities in turn imply that the pro-characteristic ideal of the $\mathcal{A}$-module $\varprojlim_n M_{(n)}$ is equal to $\mathrm{char}_{\mathcal{A}}(M)$. To complete the proof of (ii), it is now enough to note that the canonical map $M \to \varprojlim_n M_{(n)}$ is bijective as a consequence of Proposition 2.14(i) (and the first assertion of Lemma 3.1(ii)). □

Remark 3.5. The assumptions used in [2] are more general than those of Proposition 3.4. Specifically, the authors of loc. cit. assume only to be given a Krull domain Λ that arises as the inverse limit (over $d \in \mathbb{N}$) of Noetherian Krull domains Λ_d and a Λ-module M arising as the inverse limit of torsion Λ_d-modules M_d. Then, under suitable hypotheses on each Λ_d, they formulate conditions on the modules M_d that are analogous to the conclusion of Proposition 3.4(i) and, assuming these conditions to be satisfied, [2, Th. 2.13] provides a well-defined 'pro-characteristics ideal' $\widetilde{\mathrm{Ch}}_\Lambda(M)$ of M. We now assume M is a finitely-presented, torsion Λ-module that is supported on only finitely many primes in $\mathcal{P}_\Lambda$, each of which is finitely generated. Then M is also an admissible Λ-module (cf. Remarks 2.2(i) and 2.5(i)) and so has a generalised characteristic ideal $\mathrm{char}_\Lambda(M)$ in the sense of Definition 2.10. As a possible extension of Proposition 3.4 (and Proposition 2.14(iii)), it would seem reasonable to expect that in any such case $\mathrm{char}_\Lambda(M)$ should be closely related to $\widetilde{\mathrm{Ch}}_\Lambda(M)$.

3.2. **Structure results.** We henceforth fix a global function field k of characteristic p and a Galois extension K of k that is ramified at only finitely many places and such that the group $\Gamma := \mathrm{Gal}(K/k)$ is topologically isomorphic to a direct product $\mathbb{Z}_p^{\mathbb{N}} \times G$ for a finite abelian group G. We fix such an isomorphism and, in addition, a finite non-empty set of places Σ of k that contains all places that ramify in K but no place that splits completely in K. For every intermediate field L of K/k we set $\Gamma_L := \mathrm{Gal}(L/k)$ and, if L/k is finite, we write $\mathcal{O}_L^{\Sigma}$ for the subring of L comprising elements that are integral at all places outside those above Σ.

3.2.1. *Statement of the main results.* For a finite extension F of k in K, the result of [29, Chap. V, Th. 1.2] implies that the sum

$$\theta_F^{\Sigma} := [F:k]^{-1} \sum\nolimits_{\psi \in \mathrm{Hom}(\Gamma_F, \mathbb{Q}_p^{c,\times})} \sum\nolimits_{\gamma \in \Gamma_F} \psi(\gamma^{-1}) L_{\Sigma}(\psi, 0)$$

is a well-defined element of $\mathbb{Z}_p[\Gamma_F]$, where $L_{\Sigma}(\psi,0)$ denotes the value at 0 of the Σ-truncated Dirichlet L-series of ψ (here we use that, in terms of the notation of loc. cit., θ_F^{Σ} is equal to $\Theta_{\Sigma}(1)$ and, as $p = \mathrm{char}(k)$, the integer e is prime to p). In addition, the behaviour of Dirichlet L-series under inflation of characters implies the elements θ_F^{Σ} are compatible with respect to the projection maps $\mathbb{Z}_p[\Gamma_{F'}] \to \mathbb{Z}_p[\Gamma_F]$ for any finite extension F' of k in K with $F \subset F'$ and so, for each extension L of k in K, one obtains a well-defined element of $\mathbb{Z}_p[[\Gamma_L]]$ by setting

$$\theta_L^{\Sigma} := \varprojlim_{U \in \mathcal{U}(\Gamma_L)} \theta_{L^U}^{\Sigma}.$$

For each such L, we also set

$$H^1((\mathcal{O}_L^{\Sigma})_{W\text{ét}}, \mathbb{Z}_p(1)) := \varprojlim_{U \in \mathcal{U}(\Gamma_L)} \big(\mathbb{Z}_p \otimes_{\mathbb{Z}} H^1((\mathcal{O}_{L^U}^{\Sigma})_{W\text{ét}}, \mathbb{G}_m)\big)$$

and both

$$\mathrm{Pic}^0(L)_p := \varprojlim_{U \in \mathcal{U}(\Gamma_L)} (\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathrm{Pic}^0(L^U)) \quad \text{and} \quad \mathrm{Cl}(\mathcal{O}_L^{\Sigma})_p := \varprojlim_{U \in \mathcal{U}(\Gamma_L)} (\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathrm{Cl}(\mathcal{O}_{L^U}^{\Sigma})),$$

where $(-)_{W\text{ét}}$ denotes the Weil-étale site defined by Lichtenbaum in [23, §2] and $\mathrm{Pic}^0(L^U)$ the degree zero divisor class group of L^U, and the respective limits are with respect to the natural corestriction and norm maps.

We next fix a $\mathbb{Z}_p$-basis $\{\gamma_i\}_{i \in \mathbb{N}}$ of $\mathbb{Z}_p^{\mathbb{N}}$ (as at the beginning of §3.1) and, for each $n \in \mathbb{N}$, write $\Gamma(n)$ for the $\mathbb{Z}_p$-module generated by $\{\gamma_i\}_{i>n}$ and K_n for the fixed field of $\Gamma(n)$ in K (so that Γ_{K_n} is isomorphic to $\mathbb{Z}_p^n \times G$). We also write Γ_v for the decomposition group in Γ of each v in Σ and consider the following condition on K and Σ.

Hypothesis 3.6. *There exists a natural number n_0 such that, for every v in Σ, the group $\Gamma(n_0) \cap \Gamma_v$ is not open in Γ_v.*

We note that this hypothesis is satisfied in the setting of the main results of both Anglès et al [1] and Bley and Popescu [6] and hence that the structural aspects of the next result complement these earlier results (for more details see the discussions in Remarks 3.10 and 3.11 below).

We use the fixed basis $\{\gamma_i\}_{i\in\mathbb{N}}$ of $\mathbb{Z}_p^{\mathbb{N}}$ to identify the completed p-adic group ring $\mathbb{Z}_p[[\Gamma]]$ with the group ring $\mathcal{A}=\mathcal{R}[G]$ of G over the power series ring $\mathcal{R}=\mathbb{Z}_p[[\mathbb{Z}_p^{\mathbb{N}}]]$. In the sequel we shall thereby regard the inverse limit

$$M := H^1((\mathcal{O}_K^\Sigma)_{W\text{ét}},\mathbb{Z}_p(1))$$

as an $\mathcal{A}$-module.

Finally, for each n we set $\mathcal{A}_n := \mathcal{R}_n[G]\cong\mathbb{Z}_p[[\Gamma_{K_n}]]$ and $M_{(n)} := \mathcal{A}_n\otimes_{\mathcal{A}} M$.

Theorem 3.7. *The $\mathcal{A}$-module M has the following properties.*

(i) *M is quadratically-presented and, for every n, the $\mathcal{A}_n$-module $M_{(n)}$ is isomorphic to $H^1((\mathcal{O}_{K_n}^\Sigma)_{W\text{ét}},\mathbb{Z}_p(1))$.*

In the remainder of the result we assume that K and Σ satisfy Hypothesis 3.6.

(ii) *M is torsion and $\mathcal{P}_{\mathcal{A}}(M)$ is finite.*

(iii) *If $|G|$ does not belong to any prime in $\mathcal{P}_{\mathcal{A}}(M)$, then there exists a pseudo-isomorphism of $\mathcal{A}$-modules of the form*

$$M\to\bigoplus\nolimits_{\mathfrak{p}\in\mathcal{P}_{\mathcal{A}}(M)}\bigoplus\nolimits_{1\le i\le n(\mathfrak{p})}\mathcal{A}/\mathfrak{p}^{e(\mathfrak{p})_i}$$

(for suitable natural numbers $n(\mathfrak{p})$ and $e(\mathfrak{p})_i$). Setting $e(\mathfrak{p}) := \sum_{1\le i\le n(\mathfrak{p})}e(\mathfrak{p})_i$ for each $\mathfrak{p}\in\mathcal{P}_A(M)$, one also has

$$\prod\nolimits_{\mathfrak{p}\in\mathcal{P}_{\mathcal{A}}(M)}\mathfrak{p}^{e(\mathfrak{p})}\subseteq\bigcap\nolimits_{\mathfrak{q}\in\mathcal{P}_R}\big(\prod\nolimits_{\mathfrak{p}\in\mathcal{P}_{\mathcal{A}}(M)}\mathfrak{p}^{e(\mathfrak{p})}\big)_{\mathfrak{q}}=\mathcal{A}\cdot\theta_K^\Sigma, \tag{20}$$

with equality if and only if $\prod_{\mathfrak{p}\in\mathcal{P}_{\mathcal{A}}(M)}\mathfrak{p}^{e(\mathfrak{p})}$ is a principal ideal of $\mathcal{A}$.

(iv) *If $|G|$ is prime to p, then the inclusion in (20) is an equality and, in addition, for every $n\ge n_0$ the $\mathcal{A}_n$-modules*

$$H^1((\mathcal{O}_{K_{n+1}}^\Sigma)_{W\text{ét}},\mathbb{Z}_p(1))^{X_{n+1}=0}\quad\textit{and}\quad \mathrm{Cl}(\mathcal{O}_{K_{n+1}}^\Sigma)_p^{X_{n+1}=0}$$

are both pseudo-null.

This result has the following concrete consequence for the $\mathcal{A}$-module $\mathrm{Pic}^0(K)_p$.

Corollary 3.8. *Assume K and Σ satisfy Hypothesis 3.6. Then $\mathrm{Pic}^0(K)_p$ is a torsion $\mathcal{R}$-module. In addition, if $\mathrm{Pic}^0(K)_p$ is finitely generated over $\mathcal{R}$, then at most one place that ramifies in K has an open decomposition subgroup in Γ and, if such a place v exists, then one has $\Gamma_v=\Gamma$.*

The proof of these results will occupy the remainder of §3.2.

3.2.2. *Preliminaries on Weil-étale cohomology.* We first recall some general facts about Weil-étale cohomology.

For a commutative Noetherian ring Λ, we write $\mathsf{D}(\Lambda)$ for the derived category of complexes of Λ-modules and $\mathsf{D}^{\mathrm{perf}}(\Lambda)$ for the full triangulated subcategory of $\mathsf{D}(\Lambda)$ comprising complexes isomorphic to a bounded complex of finitely generated projective Λ-modules.

For a finite extension F of k in K we also write C_F for the unique geometrically irreducible smooth projective curve with function field F and j_F^Σ for the natural open immersion $\mathrm{Spec}(\mathcal{O}_F^\Sigma)\to C_F$. We then define an object of $\mathsf{D}(\mathbb{Z}_p[\Gamma_F])$ by setting

$$D^\bullet_{F,\Sigma} := \mathsf{R}\mathrm{Hom}_{\mathbb{Z}_p}(\mathsf{R}\Gamma((C_F)_{\text{ét}},j^\Sigma_{F,!}(\mathbb{Z}_p)),\mathbb{Z}_p[-2]).$$

We recall that $D^\bullet_{F,\Sigma}$ belongs to $\mathsf{D}^{\rm perf}(\mathbb{Z}_p[\Gamma_F])$ (cf. [8, Lem. 3.3]), and also that there exist canonical isomorphisms

$$\begin{aligned} H^1(D^\bullet_{F,\Sigma}) &\cong \mathbb{Z}_p \otimes_{\mathbb{Z}} H^1(\mathsf{R}\mathrm{Hom}_{\mathbb{Z}}(\mathsf{R}\Gamma((C_F)_{W\acute{e}t}, j^\Sigma_{F,!}(\mathbb{Z})), \mathbb{Z}[-2])) \\ &\cong \mathbb{Z}_p \otimes_{\mathbb{Z}} H^1((\mathcal{O}^\Sigma_F)_{W\acute{e}t}, \mathbb{G}_m) = H^1((\mathcal{O}^\Sigma_F)_{W\acute{e}t}, \mathbb{Z}_p(1)). \end{aligned} \tag{21}$$

Here the first isomorphism is a consequence of [23, Prop. 2.4(g)] and the second of the duality theorem in Weil-étale cohomology [23, Th. 5.4(a)] and the equality follows directly from our definition of $H^1((\mathcal{O}^\Sigma_F)_{W\acute{e}t}, \mathbb{Z}_p(1))$.

We next recall (from the proof of [8, Prop. 4.1]) that $D^\bullet_{F,\Sigma}$ is acyclic in degrees greater than one and such that, for each intermediate field F' of F/k, there is exists a canonical projection formula isomorphism $\mathbb{Z}_p[\Gamma_{F'}] \otimes^{\mathsf{L}}_{\mathbb{Z}_p[\Gamma_F]} D^\bullet_{F,\Sigma} \cong D^\bullet_{F',\Sigma}$ in $\mathsf{D}(\mathbb{Z}_p[\Gamma_{F'}])$. These facts combine with (21) to imply that the natural corestriction map $H^1((\mathcal{O}^\Sigma_F)_{W\acute{e}t}, \mathbb{G}_m) \to H^1((\mathcal{O}^\Sigma_{F'})_{W\acute{e}t}, \mathbb{G}_m)$ induces a canonical isomorphism of $\mathbb{Z}_p[\Gamma_{F'}]$-modules

$$\mathbb{Z}_p[\Gamma_{F'}] \otimes_{\mathbb{Z}_p[\Gamma_F]} H^1((\mathcal{O}^\Sigma_F)_{W\acute{e}t}, \mathbb{Z}_p(1)) \cong H^1((\mathcal{O}^\Sigma_{F'})_{W\acute{e}t}, \mathbb{Z}_p(1)). \tag{22}$$

Remark 3.9. We can now provide some context for Theorem 3.7 by recalling that explicit relations between the complexes $D^\bullet_{F,\Sigma}$ and leading terms of Σ-truncated Artin L-series have already been established elsewhere. In the case of finite abelian extensions F/k, these relations are obtained by the main result of Lai, Tan and the first author in [8] and in the case of arbitrary finite Galois extensions F/k by the main result of Kakde and the first author in [9].

3.2.3. *The proof of Theorem 3.7.* At the outset we fix an exhaustive separated decreasing filtration $(\Delta_n)_{n\in\mathbb{N}}$ of the subgroup $\mathbb{Z}_p^{\mathbb{N}}$ of Γ by open subgroups. We set $F_n := K^{\Delta_n}$, write J_n for the kernel of the natural projection map

$$\mathcal{A} \twoheadrightarrow \mathcal{A}_{[n]} := \mathbb{Z}_p[\Gamma_{F_n}] = \mathbb{Z}_p[\Gamma/\Delta_n] \cong \mathbb{Z}_p[(\mathbb{Z}_p^{\mathbb{N}}/\Delta_n)][G],$$

and for each $\mathcal{A}$-module N, respectively homomorphism of $\mathcal{A}$-modules θ, we set $N_{[n]} := \mathcal{A}_{[n]} \otimes_{\mathcal{A}} N$ and $\theta_{[n]} := \mathrm{id}_{\mathcal{A}_{[n]}} \otimes_{\mathcal{A}} \theta$. Then

$$J_\bullet := (J_n)_{n\in\mathbb{N}}$$

is a separated decreasing filtration with respect to which $\mathcal{A}$ is complete. In addition, the isomorphisms (22) with F/F' equal to each F_n/F_{n-1} imply the $\mathcal{A}$-module M is $J_\bullet$-complete and that, for every n, there is a natural isomorphism $M_{[n]} \cong H^1((\mathcal{O}^\Sigma_{F_n})_{W\acute{e}t}, \mathbb{Z}_p(1))$.

Turning now to the proof of Theorem 3.7, we first observe the isomorphisms in the second assertion of (i) are directly induced by the descent isomorphisms (22). We then claim that, to prove the quadratic-presentability of M (and hence complete the proof of (i)), it suffices to inductively construct, for every n, an exact commutative diagram of $\mathcal{A}_{[n]}$-modules

$$\begin{array}{ccccccc} \mathcal{A}^d_{[n]} & \xrightarrow{\theta_n} & \mathcal{A}^d_{[n]} & \xrightarrow{\pi_n} & M_{[n]} & \longrightarrow & 0 \\ {\scriptstyle\tau^0_n}\big\downarrow & & {\scriptstyle\tau^1_n}\big\downarrow & & {\scriptstyle\tau_n}\big\downarrow & & \\ \mathcal{A}^d_{[n-1]} & \xrightarrow{\theta_{n-1}} & \mathcal{A}^d_{[n-1]} & \xrightarrow{\pi_{n-1}} & M_{[n-1]} & \longrightarrow & 0 \end{array} \tag{23}$$

in which the natural number d is independent of n, all maps π_n and τ_n^0 are surjective and τ_n^1 and τ_n are the tautological projections. To justify this reduction we use the fact that Δ_{n-1}/Δ_n is a finite p-group and hence that the kernel of the projection $\mathcal{A}_{[n]} \to \mathcal{A}_{[n-1]}$ is contained in the Jacobson radical of (the finitely generated $\mathbb{Z}_p$-algebra) $\mathcal{A}_{[n]}$. This in turn implies that the natural maps $\mathrm{GL}_d(\mathcal{A}_{[n]}) \to \mathrm{GL}_d(\mathcal{A}_{[n-1]})$ are surjective and therefore, since $\mathcal{A}$ is $J_\bullet$-complete, that the inverse limit of $\mathcal{A}_{[n]}^d$ with respect to the maps τ_n^0 is isomorphic to $\mathcal{A}^d$. Then, since M is also $J_\bullet$-complete (and the inverse limit functor is exact on the category of finitely generated $\mathbb{Z}_p$-modules), by passing to the limit over n of the above diagrams one obtains an exact sequence of $\mathcal{A}$-modules

$$\mathcal{A}^d \xrightarrow{\theta} \mathcal{A}^d \xrightarrow{\pi} M \to 0 \tag{24}$$

(with $\theta = \varprojlim_n \theta_n$ and $\pi = \varprojlim_n \pi_n$) which shows directly that M is quadratically-presented.

To complete the proof of (i), we must therefore construct the diagrams (23). To do this, we note that F_1 is a finite extension of k and hence that $M_{[1]} \cong H^1((\mathcal{O}_{F_1}^\Sigma)_{W\text{ét}}, \mathbb{Z}_p(1))$ is finitely generated over $\mathcal{A}_{[1]}$ (this follows, for example, from (21) and the fact $D^\bullet_{F_1,\Sigma}$ belongs to $\mathsf{D}^{\mathrm{perf}}(\mathcal{A}_{[1]})$). We can therefore fix a natural number d and a subset $\{m_i\}_{1\le i\le d}$ of M whose image in $M_{[1]}$ generates $M_{[1]}$ over $\mathcal{A}_{[1]}$. For each n, we write $m_{i,n}$ for the projection of m_i to $M_{[n]}$. We then note that, just as above, the kernel of the projection $\mathcal{A}_{[n]} \to \mathcal{A}_{[1]}$ lies in the Jacobson radical of the (Noetherian) ring $\mathcal{A}_{[n]}$, and hence that the tautological isomorphism $\mathcal{A}_{[1]} \otimes_{\mathcal{A}_{[n]}} M_{[n]} \cong M_{[1]}$ combines with Nakayama's Lemma and our choice of elements $\{m_i\}_{1\le i\le d}$ to imply $\{m_{i,n}\}_{1\le i\le d}$ generates the $\mathcal{A}_{[n]}$-module $M_{[n]}$. We therefore obtain the right hand commutative square in (23) by defining π_n (and similarly π_{n-1}) to be the map of $\mathcal{A}_{[n]}$-modules that sends the i-th element in the standard basis of $\mathcal{A}_{[n]}^d$ to $m_{i,n}$.

By following the argument of [8, Prop. 4.1] it now follows that $D^\bullet_{F_n,\Sigma}$ can be represented by a complex of the form $P_n \xrightarrow{\theta_n} \mathcal{A}_{[n]}^d$ in which P_n is a finitely generated projective $\mathcal{A}_{[n]}$-module (placed in degree zero), $\mathrm{im}(\theta_n) = \ker(\pi_n)$ and π_n induces an isomorphism between $\mathrm{coker}(\theta_n)$ and $M_{[n]}$. Then, since $\mathcal{A}_{[n]}$ is a finite product of local rings and the $\mathcal{A}_{[n]}$-equivariant Euler characteristic of $D^\bullet_{F_n,\Sigma}$ vanishes (by Flach [16, Th. 5.1]), the $\mathcal{A}_{[n]}$-module P_n is free of rank d (and so, after changing θ_n if necessary, can be taken to be $\mathcal{A}_{[n]}^d$). In particular, if we choose both of the rows in (23) in this way, then they are exact and so the commutativity of the right hand square reduces us to proving the existence of a surjective map τ_n^0 that makes the left hand square commute. To do this we can first choose a morphism of $\mathcal{A}_{[n-1]}$-modules $\tau_n' : (\mathcal{A}_{[n]}^d)_{[n-1]} \to \mathcal{A}_{[n-1]}^d$ for which the associated diagram

$$\begin{tikzcd}
(\mathcal{A}_{[n]}^d)_{[n-1]} \arrow[r, "(\theta_n)_{[n-1]}"] \arrow[d, "\tau_n'"'] & (\mathcal{A}_{[n]}^d)_{[n-1]} \arrow[d, "\cong"', "(\tau_n^1)_{[n-1]}"] \\
\mathcal{A}_{[n-1]}^d \arrow[r, "\theta_{n-1}"] & \mathcal{A}_{[n-1]}^d
\end{tikzcd}$$

commutes and represents the canonical isomorphism $\mathcal{A}_{[n-1]} \otimes^{\mathsf{L}}_{\mathcal{A}_{[n]}} D^\bullet_{F_n,\Sigma} \cong D^\bullet_{F_{n-1},\Sigma}$. In particular, since the morphism of complexes represented by this diagram is a quasi-isomorphism and $(\tau_n^1)_{[n-1]}$ is bijective, the map τ_n' must also be bijective. The composite map

$$\tau_n^0 : \mathcal{A}^d_{[n]} \twoheadrightarrow (\mathcal{A}^d_{[n]})_{[n-1]} \xrightarrow{\tau_n'} \mathcal{A}^d_{[n-1]}$$

(in which the first map is the tautological projection) is then surjective and such that the diagram (23) commutes, as required to complete the proof of (i).

In the rest of the argument we assume that K and Σ satisfy Hypothesis 3.6.

To prove (ii) we note that, by Proposition 2.6(iii)(b), M is a torsion $\mathcal{R}$-module if and only if it is a torsion $\mathcal{A}$-module. The exact sequence (24) therefore implies that M is a torsion $\mathcal{R}$-module if and only if $\det(\theta)$ is a non-zero divisor of $\mathcal{A}$. To investigate this condition, we recall that, for each n, K_n denotes $K^{\Gamma(n)}$ and we set $\Gamma_n := \Gamma/\Gamma(n) = \mathrm{Gal}(K_n/k)$ so that $\mathcal{A}_n = \mathbb{Z}_p[[\Gamma_n]]$. We also write $I_\bullet := (I_n)_{n\in\mathbb{N}}$ for the separated decreasing filtration of $\mathcal{A}$ in which each I_n is the kernel of the natural projection map $\rho_{\langle n\rangle} : \mathcal{A} \to \mathcal{A}_n$.

Then, for every $n \geq n_0$, Hypothesis 3.6 implies that the decomposition subgroup in Γ_n of every place in Σ is infinite. Hence, for each such n, the results of [8, Prop. 4.1 and Prop. 4.4] combine to imply that $\rho_{\langle n\rangle}(\det(\theta))$ and $\theta^\Sigma_{K_n}$ are non-zero divisors of $\mathcal{A}_n$ such that

$$\mathcal{A}_n \cdot \rho_{\langle n\rangle}(\det(\theta)) = \mathcal{A}_n \cdot \theta^\Sigma_{K_n}. \tag{25}$$

This implies, in particular, that $\det(\theta) = (\rho_{\langle n\rangle}(\det(\theta)))_{n\geq n_0}$ is a non-zero divisor in the ring $\mathcal{A} = \varprojlim_n \mathcal{A}_n = \varprojlim_{n\geq n_0} \mathcal{A}_n$, and so the first assertion of (ii) is proved. In addition, the fact that $\mathcal{P}_\mathcal{A}(M)$ is finite follows directly from Lemma 3.1(i) and Proposition 2.6(iii)(c). This completes the proof of (ii).

To prove (iii), we note first that the results of (i) and (ii) combine with Lemma 3.1(ii) to imply, under the stated hypotheses, that M is a finitely-presented, admissible, torsion $\mathcal{A}$-module. From Theorem 2.3(ii)(b), we can therefore deduce the existence of a pseudo-isomorphism of $\mathcal{A}$-modules of the form

$$M \to \bigoplus\nolimits_{\mathfrak{p}\in\mathcal{P}_\mathcal{A}(M)} \bigoplus\nolimits_{1\leq i\leq n(\mathfrak{p})} \mathcal{A}/\mathfrak{p}^{e(\mathfrak{p})_i}$$

for suitable natural numbers $n(\mathfrak{p})$ and $e(\mathfrak{p})_i$. Upon setting $e(\mathfrak{p}) := \sum_{1\leq i\leq n(\mathfrak{p})} e(\mathfrak{p})_i$ and combining this pseudo-isomorphism with the explicit definition of the lower generalised characteristic ideal $\mathrm{char}_\mathcal{A}(M)$ (and the result of Proposition 2.11(i)(a)) one then obtains an equality

$$\prod\nolimits_{\mathfrak{p}\in\mathcal{P}_\mathcal{A}(M)} \mathfrak{p}^{e(\mathfrak{p})} = \mathrm{char}_\mathcal{A}(M).$$

Next we note that, as $\rho_{\langle n\rangle}(\det(\theta))$ is a non-zero divisor for each $n \geq n_0$, the equality (25) implies the existence for each such n of an element u_n of $\mathcal{A}_n^\times$ with $\rho_{\langle n\rangle}(\det(\theta)) = u_n \cdot \theta^\Sigma_{K_n}$. In particular, the family $u := (u_n)_{n\geq n_0}$ belongs to $\mathcal{A}^\times = \varprojlim_{n\geq n_0} \mathcal{A}_n^\times$ and is such that $\det(\theta) = u \cdot \theta^\Sigma_K$. From the resolution (24) one therefore has

$$\mathrm{Fit}^0_\mathcal{A}(M) = \mathcal{A} \cdot \det(\theta) = \mathcal{A} \cdot \theta^\Sigma_K.$$

Given the last two displayed equalities, all of the claims in (iii) follow directly from Proposition 2.11(i)(b).

To prove (iv) we assume $|G|$ is prime to p and adapt the argument of Proposition 3.4. Specifically, in this case every prime in $\mathcal{P}_\mathcal{A}$ is principal since $\mathcal{A}$ is a finite direct product of unique factorisation domains. The first assertion of (iv) therefore follows directly from the final assertion of (iii). To prove the remaining assertions in (iv), we note that the resolution (24) combines with the isomorphisms in (i) to imply that, for each n, the $\mathcal{A}_n$-module $\mathrm{cok}(\mathrm{id}_{\mathcal{A}_n} \otimes_{\mathcal{A}} \theta) \cong \mathcal{A}_n \otimes_{\mathcal{A}} M = M_{(n)}$ is isomorphic to $H^1((\mathcal{O}_{K_n}^{\Sigma})_{W\acute{e}t}, \mathbb{Z}_p(1))$.

In particular, if $n \geq n_0$, then the latter module is torsion since it is annihilated by the non-zero divisor $\det(\mathrm{id}_{\mathcal{A}_n} \otimes_{\mathcal{A}} \theta) = \rho_{\langle n\rangle}(\det(\theta))$ of $\mathcal{A}_n$. Given this, the pseudo-nullity of $H^1((\mathcal{O}_{K_{n+1}}^{\Sigma})_{W\acute{e}t}, \mathbb{Z}_p(1))^{X_{n+1}=0}$ follows directly from the argument of Proposition 3.4(i). The $\mathcal{A}_n$-module $\mathrm{Cl}(\mathcal{O}_{K_{n+1}}^{\Sigma})_p^{X_{n+1}=0}$ is then also pseudo-null since, after taking account of the isomorphisms (21), the exact sequence [8, (4)] (with the field K in loc. cit. taken to be K_{n+1}) gives a canonical identification of $\mathrm{Cl}(\mathcal{O}_{K_{n+1}}^{\Sigma})_p$ with a submodule of $H^1((\mathcal{O}_{K_{n+1}}^{\Sigma})_{W\acute{e}t}, \mathbb{Z}_p(1))$.

3.2.4. *The proof of Corollary 3.8.* For each subset Σ' of Σ we write $\epsilon_{\Sigma'}$ for the canonical projection map $\bigoplus_{v\in\Sigma'}\mathbb{Z}_p[[\Gamma/\Gamma_v]] \to \mathbb{Z}_p$. Then, by taking the inverse limit over n of the exact sequences [8, (4)] used above (for the fields K_{n+1}), one obtains an exact sequence of $\mathcal{A}$-modules

$$0 \to \mathrm{Cl}(\mathcal{O}_K^{\Sigma})_p \to M \to \ker(\epsilon_{\Sigma}) \to 0. \tag{26}$$

In a similar way, the corresponding limits of the exact sequences [8, (5) and (6)] combine to give an exact sequence of $\mathcal{A}$-modules

$$\ker(\epsilon_{\Sigma_{\mathrm{fin}}^K}) \to \mathrm{Pic}^0(K)_p \to \mathrm{Cl}(\mathcal{O}_K^{\Sigma})_p \to \mathbb{Z}_p/(n_K) \to 0, \tag{27}$$

in which Σ_{fin}^K is the subset of Σ comprising places that have finite residue degree in K/k and n_K is a (possibly zero) integer.

We now assume that Hypothesis 3.6 is satisfied. In this case the $\mathcal{A}$-module M is finitely-presented and torsion (by Theorem 3.7(i) and (ii)) and the $\mathcal{A}$-module $\ker(\epsilon_{\Sigma_{\mathrm{fin}}^K})$ is torsion. The first of these facts combines with the sequence (26) to imply both that the $\mathcal{A}$-module $\mathrm{Cl}(\mathcal{O}_K^{\Sigma})_p$ is torsion and also (by using the general results of [21, Th. 2.1.2, (2) and (3)]) that it is finitely generated if and only if the $\mathcal{A}$-module $\ker(\epsilon_{\Sigma})$ is finitely-presented. From the sequence (27) we can then also deduce that $\mathrm{Pic}^0(K)_p$ is a torsion $\mathcal{A}$-module (and hence a torsion $\mathcal{R}$-module) and also that $\mathrm{Cl}(\mathcal{O}_K^{\Sigma})_p$ is finitely generated (over $\mathcal{A}$) if $\mathrm{Pic}^0(K)_p$ is finitely generated over $\mathcal{R}$.

To complete the proof we now argue by contradiction and, for this, the above observations imply it is enough to assume both that $\ker(\epsilon_{\Sigma})$ is finitely-presented (over $\mathcal{A}$) and that there are either two places v_1 and v_2 in Σ such that Γ_{v_1} and Γ_{v_2} are open, or at least one place v_1 in Σ for which Γ_{v_1} is open and not equal to Γ. We then define an open subgroup of Γ by setting $\Gamma' := \Gamma_{v_1} \cap \Gamma_{v_2}$ in the first case and $\Gamma' := \Gamma_{v_1}$ in the second case, we set $\mathcal{A}' := \mathbb{Z}_p[[\Gamma']]$ and we write I and I' for the kernels of the respective canonical projection maps $\mathcal{A} \to \mathbb{Z}_p$ and $\mathcal{A}' \to \mathbb{Z}_p$.

Then the definition of Γ' ensures that the $\mathcal{A}'$-module $\ker(\epsilon_{\Sigma})$ is both finitely-presented and contains a direct summand that is isomorphic to the trivial module $\mathbb{Z}_p$. This implies (via [21, Th. 2.1.2(4)]) that $\mathbb{Z}_p$ is finitely-presented as an $\mathcal{A}'$-module and hence, by applying [21, Lem. 2.1.1] to the tautological short exact sequence

$$0 \to I' \to \mathcal{A}' \to \mathbb{Z}_p \to 0,$$

that I' is finitely generated over $\mathcal{A}'$. However, writing d for the order of Γ/Γ', there exists an exact sequence of $\mathcal{A}'$-modules

$$0 \to (I')^d \to I \to \mathbb{Z}_p^{d-1}$$

and so one can deduce that I is finitely generated over $\mathcal{A}'$, and hence also over $\mathcal{R}$. However, this last assertion is easily shown to be false and this contradiction completes the proof of Corollary 3.8.

Remark 3.10. Assume that K is a Carlitz-Hayes cyclotomic extension of k, as considered by Anglès et al in [1]. In this case $\Gamma = \mathbb{Z}_p^{\mathbb{N}}$ (so $\mathcal{A} = \mathcal{R}$) and $\Sigma = \{v\}$ with v a place that is totally ramified in K. Hence $\Gamma_v = \Gamma$ (so that Hypothesis 3.6 is clear) and, as v is totally ramified in K, for each $U \in \mathcal{U}(\Gamma)$ the integers c^U and m^U_Σ that occur in [8, (5)] are both equal to 1 and so (27) is valid with $n_K = 1$. Thus, in this case, the exact sequences (26) and (27) combine to induce identifications $M = \mathrm{Cl}(\mathcal{O}_K^\Sigma)_p = \mathrm{Pic}^0(K)_p$.

In addition, since M is quadratically-presented as an $\mathcal{R}$-module (by (24)), the results of Proposition 2.11(i)(b) (with G trivial and $R = \mathcal{R}$) and Proposition 3.4(ii) (with G trivial) imply that the generalised characteristic ideal $\mathrm{char}_{\mathcal{R}}(M)$ coincides both with $\mathrm{Fit}^0_{\mathcal{R}}(M)$ and with the pro-characteristic ideal $\widetilde{\mathrm{Ch}}_{\mathcal{R}}(M)$ of M defined in [2]. Given this, one finds that the explicit structural information concerning M that is provided by claims (iii) and (iv) of Theorem 3.7 strengthens the main results of [1] concerning $\mathrm{Pic}^0(K)_p$ (see, in particular, [1, Th. 5.2, Rem. 5.3]).

Remark 3.11. Assume that K is a Drinfeld modular tower extension L_∞ of k of the form specified by Bley and Popescu in [6, §2.2]. In this case $\mathcal{A} = \mathcal{R}[G]$ with G isomorphic to $\mathrm{Gal}(H_{\mathfrak{f}\mathfrak{p}}/k)$ for a 'real' ray class field $H_{\mathfrak{f}\mathfrak{p}}$ of k relative to a fixed prime ideal $\mathfrak{p}$ and integral ideal $\mathfrak{f}$. The set Σ therefore comprises $\mathfrak{p}$ and the set of prime divisors of $\mathfrak{f}$, and so the validity of Hypothesis 3.6 in this case follows from the argument of [6, Prop. 3.22]. We now assume that $p\mathcal{R} \notin \mathcal{P}_{\mathcal{R}}(M)$ if p divides $|G|$. Then the arguments of Proposition 2.11(i)(b) and Theorem 3.7(iii) combine to imply that the explicit ideal $\prod_{\mathfrak{p}\in\mathcal{P}_{\mathcal{A}}(M)}\mathfrak{p}^{e(\mathfrak{p})}$ that occurs as the first term in (20) is contained in $\mathrm{Fit}^0_{\mathcal{A}}(M)$, with equality if and only if it is principal (as occurs automatically if $|G|$ is prime to p). Further, by comparing the sequence (26) to the sequences of [6, (24), (25), (26)], and using the fact $\mathcal{A}_{\mathfrak{p}}$ is a discrete valuation ring for $\mathfrak{p} \in \mathcal{P}_{\mathcal{A}}(M)$, one verifies an equality of principal $\mathcal{A}$-ideals

$$\mathrm{Fit}^0_{\mathcal{A}}(M) = \mathrm{Fit}^0_{\mathcal{A}}(T_p(M_\Sigma^{(\infty)})_\Gamma).$$

Here the $\mathcal{A}$-module $T_p(M_\Sigma^{(\infty)})_\Gamma$ is (quadratically-presented and) defined in [6, §3.3] as an inverse limit $\varprojlim_n T_p(M_\Sigma^{(n)})_\Gamma$ over the p-adic Tate modules of a canonical family of Picard 1-motives. In particular, as the main result [6, Th. 1.3] (with $S = \Sigma$) of loc. cit. concerning Stickelberger elements and divisor class groups is an equality

$$\mathcal{A} \cdot \theta_K^\Sigma = \mathrm{Fit}^0_{\mathcal{A}}(T_p(M_\Sigma^{(\infty)})_\Gamma),$$

it is strengthened by the explicit structural results obtained in Theorem 3.7(iii) and (iv). Finally, we note that if $\mathfrak{p}$ decomposes in the field $H_{\mathfrak{f}\mathfrak{p}}$, then Corollary 3.8 implies that

$\mathrm{Pic}^0(L_\infty)_p$ cannot be finitely generated as an $\mathcal{R}$-module. This observation implies, in particular, that the non-splitting hypotheses on $\mathfrak{p}$ that are imposed in the results of [6, Th. 3.16 and Th. 3.17] are actually necessary for the stated conclusions to be valid.

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King's College London, Department of Mathematics, Strand, London WC2R 2LS, U.K.
Email address: david.burns@kcl.ac.uk, alex.daoud@mac.com

University College London, Department of Mathematics, 25 Gordon Street, London WC1H 0AY, U.K.
Email address: dingli.liang.20@ucl.ac.uk