ON WEIL-ÉTALE COHOMOLOGY AND ARTIN L-SERIES

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Abstract. We prove a K-theoretical leading term formula relating the Artin L-series that arise from a finite Galois covering of separated schemes over a finite field to an equivariant Euler characteristic of the compactly supported Weil-étale cohomology of $\mathbb{Z}$. We deduce from this result an explicit characterization of the Galois structure of the compactly supported Weil-étale cohomology complex of $\mathbb{Z}$ and also the proof for global function fields of a range of previously formulated conjectures including the $\Omega(3)$-Conjecture of Chinburg, the non-abelian Brumer-Stark Conjecture of Nickel and the analogue for function fields of the equivariant Tamagawa number conjecture for untwisted Tate motives. As a key step in the proof of our main result we establish several new results concerning the K-theory of group rings over Tate algebras that are of independent interest.

1. Introduction

1.1. **Statement of the main result.** Motivated by the relation between Weil groups and Galois groups, Lichtenbaum [32] introduced a new Grothendieck topology, the 'Weil-étale' topology, on the category of schemes of finite type over a finite field. He showed that, whilst Weil-étale cohomology groups share many of the same properties as étale cohomology groups, they are also in many important cases both finitely generated and related via a natural Euler characteristic construction to the leading terms of zeta functions. In this article we prove a refinement of Lichtenbaum's leading term formula in the setting of Galois coverings of schemes and then derive a variety of consequences of this result.

To give more details we fix a power q of a prime p, write $\mathbb{F}_q$ for the finite field of cardinality q and fix a separable closure $\mathbb{F}_q^c$ of $\mathbb{F}_q$. For any scheme Y' of finite type over $\mathbb{F}_q$ we write $Y'_{W\text{ét}}$ for the Weil-étale site on Y' that is defined in [32, §2]. We also write ϕ for the geometric Frobenius automorphism $x \mapsto x^{1/q}$ in $\mathrm{Gal}(\mathbb{F}_q^c/\mathbb{F}_q)$, θ for the element of $H^1(\mathrm{Spec}(\mathbb{F}_q)_{W\text{ét}}, \mathbb{Z}) = \mathrm{Hom}(\langle\phi\rangle, \mathbb{Z})$ that sends the Frobenius automorphism to 1 (and hence sends ϕ to -1) and $\theta_{Y'}$ for the pullback of θ to $H^1(Y'_{W\text{ét}}, \mathbb{Z})$. We recall that, for any sheaf $\mathcal{F}$ on $Y'_{W\text{ét}}$, taking cup product with $\theta_{Y'}$ gives a complex of the form

$$(1)\qquad 0 \to H^0(Y'_{W\text{ét}}, \mathcal{F}) \xrightarrow{\cup\theta_{Y'}} H^1(Y'_{W\text{ét}}, \mathcal{F}) \xrightarrow{\cup\theta_{Y'}} H^2(Y'_{W\text{ét}}, \mathcal{F}) \xrightarrow{\cup\theta_{Y'}} \cdots$$

For a Galois covering $f : Y \to X$ of schemes of finite type over $\mathbb{F}_q$ of group G we set

$$Z(f,t) := \sum_{\chi\in \mathrm{Ir}(G)} L^{\mathrm{Artin}}(Y,\chi,t)e_\chi.$$

Here $\mathrm{Ir}(G)$ denotes the set of irreducible complex characters of G and for each χ we write $L^{\mathrm{Artin}}(Y,\chi,t)$ for the Artin L-series defined by Milne in [33, Exam. 13.6(b)] and e_χ for the primitive central idempotent $\chi(1)|G|^{-1}\sum_{g\in G}\chi(g^{-1})g$ of $\mathbb{C}[G]$.

We recall that each function $L^{\mathrm{Artin}}(Y,\chi,t)$ is a rational function of t and we write $r_{f,\chi}$ for its order of vanishing at $t=1$. We further recall that the leading term

$$Z^*(f,1) := \sum_{\chi\in\mathrm{Ir}(G)} \lim_{t\to 1}(1-t)^{-r_{f,\chi}} L^{\mathrm{Artin}}(Y,\chi,t)e_\chi$$

of $Z(f,t)$ at $t=1$ is a well-defined element of $\zeta(\mathbb{Q}[G])^\times$, where we write $\zeta(A)$ for the centre of any ring A.

In the sequel all G-modules are to be understood, unless explicitly stated otherwise, as left modules. We then write $D(\mathbb{Z}[G])$ for the derived category of G-modules and $D^{\mathrm{perf}}(\mathbb{Z}[G])$ for the full triangulated subcategory of $D(\mathbb{Z}[G])$ comprising complexes that are 'perfect' (that is, isomorphic in $D(\mathbb{Z}[G])$ to a bounded complex of finitely generated projective G-modules). We write $K_0(\mathbb{Z}[G],\mathbb{Q}[G])$ for the relative algebraic K_0-group of the natural ring inclusion $\mathbb{Z}[G]\to\mathbb{Q}[G]$. We also use the canonical extended boundary homomorphism

$$\delta_G : \zeta(\mathbb{Q}[G])^\times \to K_0(\mathbb{Z}[G],\mathbb{Q}[G])$$

of relative algebraic K-theory and the 'refined Euler characteristic' $\chi_G(-,-)$ construction that associates to every rationally trivialized complex in $D^{\mathrm{perf}}(\mathbb{Z}[G])$ a canonical element of $K_0(\mathbb{Z}[G],\mathbb{Q}[G])$. (These constructions are both recalled in §3.1.)

We can now state the main result of this article.

Theorem 1.1. *Let*

$$\begin{array}{ccc} Y & \xrightarrow{j_Y} & Y' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ X & \xrightarrow{j} & X' \end{array} \tag{2}$$

be a Cartesian diagram in which f' is a finite Galois covering of group G, X is geometrically connected, X' and Y' are proper and of finite type over $\mathbb{F}_q$, j and j_Y are open immersions and the following two conditions are satisfied:

- (i) *the cohomology groups $H^i(Y'_{W\acute{e}t}, j_{Y,!}\mathbb{Z})$ are finitely generated in all degrees i and vanish for almost all i, and*
- (ii) *the complex (1) with $\mathcal{F}=j_{Y,!}\mathbb{Z}$ has finite cohomology groups.*

Then the complex $R\Gamma(Y'_{W\acute{e}t}, j_{Y,!}\mathbb{Z})$ belongs to $D^{\mathrm{perf}}(\mathbb{Z}[G])$ and in $K_0(\mathbb{Z}[G],\mathbb{Q}[G])$ one has

$$\delta_G(Z^*(f,1)) = -\chi_G(R\Gamma(Y'_{W\acute{e}t}, j_{Y,!}\mathbb{Z}), \epsilon_{f,j}) \tag{3}$$

where $\epsilon_{f,j}$ denotes the exact sequence of $\mathbb{Q}[G]$-modules that is induced (under condition (ii)) by the complex (1) with $\mathcal{F}=j_{Y,!}\mathbb{Z}$.

Remark 1.2. For each prime ℓ, the long exact sequence of relative K-theory of the inclusion $\mathbb{Z}_\ell[G]\subseteq\mathbb{Q}_\ell[G]$ implies that (3) determines the image of $Z^*(f,1)$ in $\zeta(\mathbb{Q}_\ell[G])^\times$ modulo the subgroup $\mathrm{Nrd}_{\mathbb{Q}_\ell[G]}(K_1(\mathbb{Z}_\ell[G]))$. In view of the explicit description of $\mathrm{Nrd}_{\mathbb{Q}_\ell[G]}(K_1(\mathbb{Z}_\ell[G]))$ that is obtained by the second author in [30], or equivalently by the methods of Ritter and

Weiss in [40], this means that (3) incorporates families of congruence relations between the leading terms at $t = 1$ of the Artin L-series $L^{\mathrm{Artin}}(Y, \chi, t)$ for varying χ in $\mathrm{Ir}(G)$.

Remark 1.3. Upon specialising (3) to the case $Y' = X'$ (so $Y = X$ and G is trivial) one recovers the leading term formula proved by Lichtenbaum in [32, Th. 7.4]. The latter result also shows that (for any G) the conditions (i) and (ii) in Theorem 1.1 are valid if $X = X'$ (so j is the identity) and Y' is both smooth and projective, or if Y' is either a curve or a smooth surface. In general, however, if Y' is either non-smooth or non-proper, then condition (i) is not always satisfied and the validity of condition (ii) is related to Tate's conjecture on the bijectivity of the cycle-class map. In a more general setting it would seem natural to ask if our methods can prove a result similar to Theorem 1.1 for the theory of 'arithmetic cohomology' introduced by Geisser in [23].

As an important step in the proof of Theorem 1.1 we shall establish several new results concerning the Whitehead groups of group rings over Tate algebras (see, in particular Theorem 2.1 and Corollary 2.3).

These results extend earlier work of several authors (for example, to the context of ramified coefficient rings) and are likely to be of independent interest. In particular, they extend the main K-theoretical results that are obtained by the second author in [30], by Ritter and Weiss in [40] and by Witte in [50] and have played a key role in the construction of 'p-adic L-functions' in various contexts of non-commutative Iwasawa theory.

In addition to these K-theoretical results, the proof of Theorem 1.1 will also make crucial use of Grothendieck's description of the Zeta function of ℓ-adic sheaves on X in terms of compactly supported ℓ-adic cohomology, of the analogous description of the Zeta function of p-adic sheaves on X in terms of crystalline cohomology that is due to Emerton and Kisin and of an explicit computation of certain Bockstein homomorphisms that arise naturally in 'descent-type' computations.

This argument demonstrates a key advantage of the sort of K-theoretical techniques developed here and, in the setting of the geometric Birch and Swinnerton-Dyer Conjecture, by Kim and the present authors [7]. Specifically, these techniques can be sufficient to study equivariant leading term conjectures for varieties over function fields and hence avoid the need to develop an appropriate formalism of (non-commutative) Iwasawa theory. This is important since, for example, there are still no ideas on how to formulate a main conjecture of Iwasawa theory for semistable abelian varieties relative to any general class of ramified p-adic Lie extensions of global function fields.

1.2. **Statement of some consequences.** We quickly review several concrete consequences that follow from Theorem 1.1.

1.2.1. To state the first we write $K_0(\mathbb{Z}[G])$ for the Grothendieck group of the category of finitely generated projective G-modules. We recall that the torsion subgroup of $K_0(\mathbb{Z}[G])$ identifies with the reduced projective class group $\mathrm{Cl}(\mathbb{Z}[G])$ of G and that to any object C of $D^{\mathrm{perf}}(\mathbb{Z}[G])$ one can associate a canonical Euler characteristic $\chi_G(C)$ in $K_0(\mathbb{Z}[G])$.

We write $\mathbb{Q}^c$ for the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$, $\mathrm{Ir}(G)$ for the set of irreducible $\mathbb{Q}^c$-valued characters of G and $\mathrm{Map}_{\mathbb{Q}}(\mathrm{Ir}(G), J(\mathbb{Q}^c))$ for the group of $\mathrm{Gal}(\mathbb{Q}^c/\mathbb{Q})$-equivariant maps from $\mathrm{Ir}(G)$ to the idele group $J(\mathbb{Q}^c)$ of $\mathbb{Q}^c$. We recall (from, for example, [21, Chap. I, Th. 2]) that Fröhlich has constructed a canonical (surjective) 'Hom-description' homomorphism

$$h_G : \mathrm{Map}_{\mathbb{Q}}(\mathrm{Ir}(G), J(\mathbb{Q}^c)) \to \mathrm{Cl}(\mathbb{Z}[G]).$$

We now fix a Cartesian diagram as in (2). Then for each character χ in $\mathrm{Ir}(G)$ we obtain a non-zero element of $\mathbb{Q}^c$ by setting

$$L^{\mathrm{Artin},*}(Y,\chi,1) := \lim_{t\to 1}(1-t)^{-r_{f,\chi}} L^{\mathrm{Artin}}(Y,\chi,t).$$

We can therefore define an element $z^{\mathrm{Artin}}(f)$ of $\mathrm{Map}(\mathrm{Ir}(G), J(\mathbb{Q}^c)))$ by setting

$$(4) \qquad z^{\mathrm{Artin}}(f)(\chi)_v := \begin{cases} j_v\big(L^{\mathrm{Artin},*}(Y,\chi,1)\big), & \text{if } \chi \text{ is symplectic and } v \text{ is archimedean,} \\ 1, & \text{otherwise} \end{cases}$$

for each χ in $\mathrm{Ir}(G)$ and each place v of $\mathbb{Q}^c$, where in each case j_v is a choice of embedding $\mathbb{Q}^c \to \mathbb{C}$ that corresponds to v (and is made precise in §6.2.2). We also note that if χ is symplectic, then $L^{\mathrm{Artin},*}(Y,\chi,1)$ is a real number.

Then the following result is a precise analogue for Weil-étale cohomology of the main result of Chinburg in [14] concerning relations between the Galois structure of de Rham cohomology complexes and Artin root numbers that are associated to tamely ramified Galois covers of schemes over $\mathbb{F}_q$.

Corollary 1.4. *Fix a Cartesian diagram as in Theorem 1.1. Then the map* $z^{\mathrm{Artin}}(f)$ *is* $\mathrm{Gal}(\mathbb{Q}^c/\mathbb{Q})$*-equivariant and in* $K_0(\mathbb{Z}[G])$ *one has*

$$\chi_G\big(R\Gamma(Y'_{W\text{ét}}, j_{Y,!}\mathbb{Z})\big) = h_G(z^{\mathrm{Artin}}(f)).$$

Hence, the element $\chi_G\big(R\Gamma(Y'_{W\text{ét}}, j_{Y,!}\mathbb{Z})\big)$ *depends only on the sign of* $L^{\mathrm{Artin},*}(Y,\chi,1)$ *for irreducible complex symplectic characters* χ *of* G *and has order dividing two.*

In particular, if $L^{\mathrm{Artin},*}(Y,\chi,1)$ *is positive for all such* χ*, then* $R\Gamma(Y'_{W\text{ét}}, j_{Y,!}\mathbb{Z})$ *is isomorphic in* $D(\mathbb{Z}[G])$ *to a bounded complex of finitely generated free* G*-modules.*

The proof of this result will be given in §6.2.

1.2.2. We now review a selection of the results that will be derived from Theorem 1.1 in the special case that X is an affine curve. Before stating the first of these we recall that, for any finite Galois extension K/k of global function fields, the 'equivariant leading term conjecture' formulated by the first author in [2, Conj. C(K/k)] is the natural function field analogue of the equivariant Tamagawa number conjecture for the untwisted Tate motive over Galois extensions of number fields (for more details see [2, Rem. 2, Rem. 3]).

The following result will be stated precisely as Theorem 7.1 and proved in §7.2.

Theorem 1.5. *For any finite Galois extension* K/k *of global function fields the equivariant leading term conjecture of* [2, Conj. C(K/k)] *is valid.*

Whilst the main result of [3] verifies the latter conjecture for all abelian extensions, the conjecture has not before been verified for *any* non-abelian Galois extensions K/k of degree divisible by the characteristic of k. In fact, Theorem 1.5 also now gives an alternative proof of the main result of [3] in that it avoids any use of Iwasawa theory.

Before stating the next result we recall that the Brumer-Stark Conjecture for global function fields concerns the explicit Galois structure of divisor class groups in finite abelian extensions. The significance of this conjecture was first observed by Mazur and it was partially proved by Tate and then proved in full by Deligne by using the theory of 1-motives (for an exposition of this proof, due to Tate, see [46, Chap. V]) and then later by Hayes by using the theory of rank one sign-normalized Drinfeld modules [28]. The 'non-abelian Brumer-Stark Conjecture' was formulated by Nickel [36] and constitutes in the case of global function fields a natural conjectural extension of the theorem of Deligne and Hayes to the setting of arbitrary Galois extensions.

Corollary 1.6. *The non-abelian Brumer-Stark Conjecture is valid for global function fields.*

In Theorem 7.7 we will actually prove a stronger form of Nickel's conjecture in which annihilators are replaced by (non-abelian) Fitting invariants. By a more detailed analysis of the proof of Theorem 7.7, we are also able to show Theorem 1.5 implies both an extended version of the original Brumer-Stark Conjecture in which leading terms (rather than values) of Artin L-series give rise to elements of higher Fitting ideals of divisor class groups and also the validity for global function fields of a refinement of the main conjecture that was formulated in [4] (see Corollary 7.11 and, for context, Remark 7.12).

Finally, we recall that the '$\Omega(3)$-Conjecture' of Chinburg is a central conjecture of classical Galois module theory. It was formulated in [11] in the setting of number fields and then later in [10] in the setting of arbitrary global fields. Hitherto, this conjecture has not been verified for *any* non-abelian Galois extension of global function fields of degree divisible by the residue characteristic. Nevertheless, in §7.4.4 we are able to combine Theorem 1.5 with earlier work of Flach and the first author in order to deduce the following result.

Corollary 1.7. *Chinburg's $\Omega(3)$-Conjecture is valid for global function fields.*

Acknowledgements We are very grateful to Wenhan Zhang for his careful reading of this article and to the anonymous referee for several helpful comments. In addition, the second named author is supported by DST-SERB grant SB/SJF/2020-21/11, SERB SUPRA grant SPR/2019/ 000422, DST FIST program - 2021 [TPN - 700661], and Tata Education and Development Trust.

Part I: General Results

In the next four sections we will establish several general results (including, perhaps most notably, Theorems 2.1 and 4.3) that will later play a key role in the proof of Theorem 1.1.

In the sequel all modules are to be understood as left modules unless explicitly stated otherwise.

For any associative, unital, left noetherian ring R we write $C^-(R)$ and $C^{\rm perf}(R)$ for the categories of bounded above complexes of projective R-modules and of bounded complexes of finitely generated projective R-modules respectively.

We also write $D(R)$ for the derived category of R-modules and $D^-(R)$ and $D^{\rm perf}(R)$ for the full triangulated subcategories of $D(R)$ comprising complexes that are respectively isomorphic to an object of $C^-(R)$ and $C^{\rm perf}(R)$.

2. Whitehead groups for group rings over Tate algebras

2.1. **Mackey functors.** In this section we use the basic theory of Mackey functors, Green rings and Green modules, the necessary background for which can be found, for example, in [38, Chap. 11].

If $\mathcal{M}$ is a Mackey functor, then for each monomorphism of finite groups $V \hookrightarrow U$ we write

$$(i_V^U)^* : \mathcal{M}(U) \to \mathcal{M}(V) \quad \text{and} \quad (i_V^U)_* : \mathcal{M}(V) \to \mathcal{M}(U)$$

for the induced group homomorphisms (in particular, we caution the reader that, for simplicity, we do not indicate the dependence of these maps on $\mathcal{M}$).

We write U^{ab} for the abelianization of a group U. Then for certain $\mathcal{M}$ the tautological surjection $U \to U^{\mathrm{ab}}$ also induces maps $\mathcal{M}(U) \to \mathcal{M}(U^{\mathrm{ab}})$ and $\mathcal{M}(U^{\mathrm{ab}}) \to \mathcal{M}(U)$ and, in any such case, we denote these maps by $(i_{U^{\mathrm{ab}}}^U)_*$ and $(i_{U^{\mathrm{ab}}}^U)^*$ respectively.

Fix a finite group $\mathcal{G}$ and a prime p. Then we write $G_0^{\mathbb{Z}_p}(\mathbb{Z}_p[\mathcal{G}])$ for the Grothendieck group of the category of finitely generated $\mathbb{Z}_p[\mathcal{G}]$-modules that are free as $\mathbb{Z}_p$-modules and recall that there are natural isomorphisms

$$G_0^{\mathbb{Z}_p}(\mathbb{Z}_p[\mathcal{G}]) \cong G_0(\mathbb{Q}_p[\mathcal{G}]) \cong K_0(\mathbb{Q}_p[\mathcal{G}])$$

(cf. [1, Chap. XI, Th. 5.3, Chap. IX, 2.4] and [42, Prop. 9]). We regard these isomorphisms as identifications and always describe elements of these groups via $K_0(\mathbb{Q}_p[\mathcal{G}])$ as this is canonically isomorphic to the free abelian group on the set of irreducible $\mathbb{Q}_p$-valued characters of $\mathcal{G}$.

We fix an algebraic closure $\mathbb{Q}_p^c$ of $\mathbb{Q}_p$ and write $R(\mathcal{G})$ for the free abelian group on the set of irreducible $\mathbb{Q}_p^c$-representations of $\mathcal{G}$.

We write $S(\mathcal{G})$ and $C(\mathcal{G})$ for the sets of all subgroups and of all cyclic subgroups of $\mathcal{G}$ respectively.

2.2. **Statement of the main algebraic theorem.** We fix a finite group G and an arbitrary prime p. We write O for the valuation ring of some fixed finite extension of $\mathbb{Q}_p$ in $\mathbb{Q}_p^c$, fix a uniformiser π of O and denote the residue field $O/(\pi)$ of O by k. We also fix an indeterminate t and recall that the Tate algebra $O\langle t\rangle$ of O relative to t is the subalgebra of the power series ring $O[[t]]$ comprising series $\sum_{i\geq 0} a_i t^i$ with the property that the sequence $(a_i)_i$ of elements of O converges to 0 as $i \to \infty$.

Throughout this section Λ will denote either O, $O\langle t\rangle$ or $O[[t]]$, and we shall consider the group ring $\Lambda[G]$.

We write $SK_1(\Lambda[G])$ for the kernel of the scalar extension map

$$K_1(\Lambda[G]) \to K_1(\Lambda\big[1/p\big][G])$$

and set

$$K_1'(\Lambda[G]) := K_1(\Lambda[G])/SK_1(\Lambda[G]).$$

We will make important use of the fact that $K_1'(\Lambda[G])$ is a Green module over the Green ring $K_0(\mathbb{Q}_p[G])$. (Note [38, Th. 11.2] states this result for the case $\Lambda = O$ but that, as per [16, Th. 4.6(i)], this restriction on coefficients is not necessary).

We set $\Lambda_{\mathbb{Q}_p^c} := \Lambda \otimes_O \mathbb{Q}_p^c$ and recall that there is a natural 'determinant' homomorphism

$$D_{\Lambda[G]} : K_1(\Lambda[G]) \to \mathrm{Hom}(R(G), \Lambda_{\mathbb{Q}_p^c}^\times)$$

with $\ker(D_{\Lambda[G]}) = SK_1(\Lambda[G])$. Via these homomorphisms the natural inclusions

$$\mathrm{Hom}(R(G), \mathbb{Q}_p^{c,\times}) \hookrightarrow \mathrm{Hom}(R(G), (O\langle t\rangle \otimes_O \mathbb{Q}_p^c)^\times) \hookrightarrow \mathrm{Hom}(R(G), (O[[t]] \otimes_O \mathbb{Q}_p^c)^\times)$$

induce canonical injective homomorphisms

$$K_1'(O[G]) \hookrightarrow K_1'(O\langle t\rangle[G]) \hookrightarrow K_1'(O[[t]][G]) \tag{5}$$

and we shall always regard these maps as inclusions.

For the trivial subgroup $U = \{1\}$ of G we write η_U for its trivial character. Then for each cyclic subgroup U of G we recursively define an element of $K_0(\mathbb{Q}_p[U])$ by setting

$$\eta_U := |U| - \sum_{U' \subsetneq U} (i^U_{U'})_*(\eta_{U'}), \tag{6}$$

where $|U|$ denotes $|U|$ times the trivial character of U.

We consider the diagonal homomorphism

$$\theta_{\Lambda[G]} : K_1'(\Lambda[G]) \xrightarrow{(\theta_U)_U} \prod_{U \in S(G)} \Lambda[U^{\mathrm{ab}}]^\times$$

that uses for each subgroup U of G the composite homomorphism

$$\theta_U : K_1'(\Lambda[G]) \xrightarrow{(i^G_U)^*} K_1'(\Lambda[U]) \xrightarrow{(i^U_{U^{\mathrm{ab}}})_*} K_1'(\Lambda[U^{\mathrm{ab}}]) \cong \Lambda[U^{\mathrm{ab}}]^\times.$$

We can now state the main result of this section.

Theorem 2.1. *The map $\theta_{\Lambda[G]}$ induces an isomorphism between $K_1'(\Lambda[G])$ and the set of all elements $(x_U)_U$ in $\prod_{U \in S(G)} \Lambda[U^{\mathrm{ab}}]^\times$ that satisfy both of the following conditions:*

($\mathrm{F}_{\Lambda[G]}$) *For every element $(\chi_U)_U$ of $\prod_{U \in S(G)} R(U^{\mathrm{ab}})$ that satisfies*

$$\sum_{U \in S(G)} (i^G_U)_* \circ (i^U_{U^{\mathrm{ab}}})^*(\chi_U) = 0$$

in $R(G)$ one has

$$\prod_{U \in S(G)} D_{\Lambda[U]}(x_U)(\chi_U) = 1.$$

($\mathrm{C}_{\Lambda[G]}$) *There exists an element x of $K_1'(\Lambda[G])$ with*

$$\prod_{U \in C(G)} (i^G_U)_*(\eta_U \cdot x_U) = x^{|G|}.$$

Remark 2.2. The condition ($\mathrm{F}_{\Lambda[G]}$) is an elementary compatibility relation and the condition ($\mathrm{C}_{\Lambda[G]}$) is also directly implied by the basic properties of Mackey functors (for more details see the proof of Proposition 2.6 below). It is thus striking that these conditions are together sufficient to characterize the image of $\theta_{\Lambda[G]}$. In this regard, we further note that the explicit description of $K_1'(\Lambda[G])$ given in Theorem 2.1 extends the K-theoretical results that have played a key role in the construction of 'p-adic L-functions' in the setting of non-commutative Iwasawa theory (cf. [48]) and so is of some independent interest.

The following result will be derived as a straightforward consequence of Theorem 2.1 and plays a key role in later arguments.

Corollary 2.3. *An element x of $K_1'(O[[t]][G])$ belongs to $K_1'(O\langle t\rangle[G])$ if and only if for all subgroups U of G one has $\theta_U(x)\in O\langle t\rangle[U^{\rm ab}]^\times$.*

The proofs of Theorem 2.1 and Corollary 2.3 will occupy §2.3, §2.4 and §2.5.

Remark 2.4. In [50, §3] Witte uses methods developed by Chinburg, Pappas and Taylor in [15, 16] to study $K_1(O\langle t\rangle[G])$ in the case that O is an unramified extension of $\mathbb{Z}_p$. In this special case his approach can be used to give a different proof of Corollary 2.3 but does not give an analogue of the finer result in Theorem 2.1.

2.3. **First observations on the proof of Theorem 2.1.**

Proposition 2.5. *The homomorphism $\theta_{\Lambda[G]}$ is injective.*

Proof. This result follows directly from the commutative diagram

$$\begin{array}{ccc} K_1'(\Lambda[G]) & \xrightarrow{D_{\Lambda[G]}} & \mathrm{Hom}(R(G),\Lambda^\times_{\mathbb{Q}_p^c}) \\ {\scriptstyle\theta_{\Lambda[G]}}\downarrow & & \downarrow{\scriptstyle\prod_{U\in S(G)}(i^U_{U^{\rm ab}})_*\circ(i^G_U)^*} \\ \prod_{U\in S(G)}\Lambda[U^{\rm ab}]^\times & \xrightarrow{(D_{\Lambda[U^{\rm ab}]})_U} & \prod_{U\in S(G)}\mathrm{Hom}(R(U^{\rm ab}),\Lambda^\times_{\mathbb{Q}_p^c}). \end{array}$$

and the fact that Brauer's induction theorem (as stated in [42, Th. 20]) implies that the right hand vertical arrow in the diagram is injective. □

Proposition 2.6. *Any element in the image of $\theta_{\Lambda[G]}$ satisfies both of the conditions ($F_{\Lambda[G]}$) and ($C_{\Lambda[G]}$).*

Proof. Fix x in $K_1'(\Lambda[G])$ and set $(x_U):=\theta_{\Lambda[G]}(x)$.

To prove (x_U) satisfies ($\mathrm{F}_{\Lambda[G]}$) we use the fact that each map θ_U is equal to the composite left hand vertical arrow in the following commutative diagram

$$\begin{array}{ccc} K_1'(\Lambda[G]) & \xrightarrow{D_{\Lambda[G]}} & \mathrm{Hom}(R(G),\Lambda^\times_{\mathbb{Q}_p^c}) \\ {\scriptstyle(i^G_U)^*}\downarrow & & \downarrow{\scriptstyle(i^G_U)^*} \\ K_1'(\Lambda[U]) & \xrightarrow{D_{\Lambda[U]}} & \mathrm{Hom}(R(U),\Lambda^\times_{\mathbb{Q}_p^c}) \\ {\scriptstyle(i^U_{U^{\rm ab}})_*}\downarrow & & \downarrow{\scriptstyle(i^U_{U^{\rm ab}})_*} \\ \Lambda[U^{\rm ab}]^\times & \xrightarrow{D_{\Lambda[U^{\rm ab}]}} & \mathrm{Hom}(R(U^{\rm ab}),\Lambda^\times_{\mathbb{Q}_p^c}). \end{array} \tag{7}$$

In particular, if (χ_U) in $\prod_{U\in S(G)}R(U^{\rm ab})$ satisfies $\sum_{U\in S(G)}(i^G_U)_*\circ(i^U_{U^{\rm ab}})^*(\chi_U)=0$, then this diagram implies that

$$\begin{aligned}\prod_{U\in S(G)} D_{\Lambda[U]}(x_U)(\chi_U) &= \prod_{U\in S(G)} D_{\Lambda[G]}(x)((i_U^G)_*\circ(i^U_{U^{\mathrm{ab}}})^*(\chi_U))\\ &= D_{\Lambda[G]}(x)\big(\textstyle\sum_{U\in S(G)}(i_U^G)_*\circ(i^U_{U^{\mathrm{ab}}})^*(\chi_U)\big)\\ &= D_{\Lambda[G]}(x)(0)\\ &= 1,\end{aligned}$$

and hence that (x_U) satisfies $(\mathrm{F}_{\Lambda[G]})$.

To prove (x_U) satisfies $(\mathrm{C}_{\Lambda[G]})$ we use the fact that for any subgroup W of G there is an equality in $K_0(\mathbb{Q}_p[W])$ of the form

$$\sum_{U\in C(W)} (i_U^W)_*(\eta_U) = |W| \tag{8}$$

(by Serre [42, Prop. 28, 27]). Taken in conjunction with the Frobenius axiom, this equality implies that

$$\begin{aligned}\prod_{U\in C(G)} (i_U^G)_*(\eta_U\cdot x_U) &= \prod_{U\in C(G)} \Big((i_U^G)_*(\eta_U)\cdot x\Big)\\ &= (\textstyle\sum_{U\in C(G)}(i_U^G)_*(\eta_U))\cdot x\\ &= |G|\cdot x\\ &= x^{|G|}\end{aligned}$$

and hence that (x_U) satisfies $(\mathrm{C}_{\Lambda[G]})$, as required. □

2.4. **Torsion considerations.** As further preparation for the proof of Theorem 2.1, in this section we investigate the torsion subgroup of $K_1'(\Lambda[G])$.

In the sequel we write A_{tor} for the torsion subgroup of an abelian group A. We also write $A_{[p]}$ and $A_{[p']}$ for the maximal subgroups of A_{tor} comprising elements whose orders are respectively a power of p and coprime to p.

2.4.1. *Preliminary results.* The first key step in the proof of Theorem 2.1 will be to prove the following useful result.

Proposition 2.7.

(i) *The kernel of the homomorphism*

$$K_1'(O[[t]][G])/K_1'(\Lambda[G]) \to \prod_{U\in S(G)} O[[t]][U^{\mathrm{ab}}]^\times/\Lambda[U^{\mathrm{ab}}]^\times$$

that is induced by $\theta_{O[[t]][G]}$ is torsion-free.

(ii) *The inclusions (5) restrict to give an equality $K_1'(O[G])_{\mathrm{tor}} = K_1'(\Lambda[G])_{\mathrm{tor}}$.*

However, before proving this result, we must first record several technical results that are obtained by a routine extension of arguments of Oliver.

Proposition 2.8. *Let P be a finite group of p-power order and z a central element of P that has order p. Then there exists a natural exact sequence of abelian groups of the form*

$$1 \to \langle z\rangle \to K_1(\Lambda[P],(1-z)\Lambda[P]) \to H_0(P,(1-z)\Lambda[P]) \xrightarrow{\omega} \mathbb{F}_p \to 1.$$

Proof. This results generalises [38, Prop. 6.4] and our argument closely follows that in loc. cit.

We set $I := (1-z)\Lambda[P]$ and write J for the Jacobson radical of $\Lambda[P]$. Then one checks easily that $I^p \subseteq p(1-z)\Lambda[P]$ and this fact combines with the argument of [38, Th. 2.8] to show that the classical p-adic logarithm gives rise to a diagram of group homomorphisms

$$\begin{array}{ccccccccc}
 & & K_1(\Lambda[P],IJ) & \longrightarrow & K_1(\Lambda[P],I) & \longrightarrow & K_1(\Lambda[P]/IJ,I/IJ) & \longrightarrow & 1 \\
 & & \cong\big\downarrow \log^{IJ} & & \big\downarrow \log^{I} & & \big\downarrow \log_0 & & \\
0 & \longrightarrow & H_0(P,IJ) & \longrightarrow & H_0(P,I) & \longrightarrow & H_0(P,I/IJ) & \longrightarrow & 0.
\end{array}$$

In this diagram $\log^{IJ}$ is bijective, both rows are exact, the first square commutes and $\log_0$ is defined to be the unique map which makes the second square commute.

Moreover, the same argument as in [38, Prop. 6.4] shows that there are natural isomorphisms of groups

$$K_1(\Lambda[P]/IJ,I/IJ) \cong \Lambda[P]/J \cong k \cong H_0(P,I/IJ),$$

with respect to which $\log_0$ identifies with the endomorphism $1-\varphi$ of k, where φ is the Frobenius map.

Given these facts, the claimed exact sequence (with the third arrow equal to $\log^I$) is derived from the above diagram by simply mimicking the argument given in loc. cit. □

Proposition 2.9. *Let P be a finite group of p-power order. Let P' be a subgroup of P generated by central commutators of order p. Write I for the kernel of the natural projection $\Lambda[P] \to \Lambda[P/P']$ and $K_1'(\Lambda[P],I)$ for the image of $K_1(\Lambda[P],I)$ in $K_1'(\Lambda[P])$. Then the image of $K_1'(\Lambda[P],I)$ under the logarithm map lies in $H_0(P,\Lambda[P])$ and coincides with the image of $H_0(P,I)$.*

Proof. Let $z_1,\dots,z_k$ be a set of central commutators that generate P'. Then one has

$$I = (z_1-1)\Lambda[P] + \cdots + (z_k-1)\Lambda[P].$$

Hence, by [38, Lem. 6.3(i)], one has $I^p \subset pI$ and therefore the logarithm map takes $K_1'(\Lambda[P],I)$ into $H_0(P,\Lambda[P])$.

Next we prove that the image of $K_1'(\Lambda[P],I)$ in $H_0(P,\Lambda[P])$ coincides with the image of $H_0(P,I)$.

To prove this in the case $k=1$ we need only use the result of Proposition 2.8. Indeed, since for any $g \in P$ the image of $(1-z_1)g$ under the map ω that occurs in the sequence in the latter result is equal to 1, the required claim follows from the exactness of this sequence and the fact that if $z_1 = hgh^{-1}g^{-1}$, then the image of $(1-z_1)g$ in $H_0(P,\Lambda[P])$ is trivial.

To prove the result in the general case we denote the image of $H_0(P,I)$ in $H_0(P,\Lambda[P])$ by $\overline{H}_0$. Then there is a natural commutative diagram

$$\begin{array}{ccc}
\bigoplus_{i=1}^{k} K_1(\Lambda[P], (1-z_i)\Lambda[P]) & \longrightarrow & \bigoplus_{i=1}^{k} H_0(P, (1-z_i)\Lambda[P]) \\
\downarrow & & \downarrow \\
K_1'(\Lambda[P], I) & \longrightarrow & \overline{H}_0
\end{array}$$

in which the vertical arrows are surjective and, via this diagram, the claimed result is easily reduced to the case $k = 1$. □

Lemma 2.10. *Let P be a finite group of p-power order. Let P' be a subgroup of P generated by central commutators of order p. Let Λ denote either O or $O[[t]]$. Then the map $SK_1(\Lambda[P]) \to SK_1(\Lambda[P/P'])$ that is induced by the natural surjection $\Lambda[P] \to \Lambda[P/P']$ is itself surjective.*

Proof. In the case $\Lambda = O$ this result is from [38, p. 277, (3)] and so follows from [38, Prop 8.1] and [38, Thm 8.7].

To deal with the case $\Lambda = O[[t]]$, we use the fact that for any finite group Q of p-power order there is a natural isomorphism $\Lambda[Q] \cong \varprojlim_n O[\mathbb{Z}/p^n \times Q]$. In particular, since by [38, Prop 8.12] there are natural isomorphisms $SK_1(O[\mathbb{Z}/p^n \times Q]) \cong SK_1(O[Q])$, the groups $SK_1(\Lambda[Q])$ and $SK_1(O[Q])$ are naturally isomorphic and so the claimed result for $\Lambda = O[[t]]$ follows directly from the result for $\Lambda = O$. □

The next result generalises [38, Th. 12.3].

Proposition 2.11. *Assume that the quotient field L of O is a Galois extension of $\mathbb{Q}_p$. Let P be a finite p-group for which there exists a homomorphism $\tau : P \to \mathrm{Gal}(L/\mathbb{Q}_p)$ such that $L/L^{\tau(P)}$ is unramified, and set $P' := \ker(\tau)$.*

Then, with Λ denoting either O or $O[[t]]$ and $\Lambda[P]^\tau$ the corresponding twisted group ring associated to τ, the transfer homomorphism $K_1'(\Lambda[P]^\tau) \to H^0(P/P', K_1'(\Lambda[P']))$ that is induced by the inclusion $\Lambda[P'] \subseteq \Lambda[P]^\tau$ is bijective.

Proof. We note first that the argument used to prove [38, Lem. 12.1] shows that the functorial homomorphism $K_1'(\Lambda[P']) \to K_1'(\Lambda[P]^\tau)$ induced by the inclusion $\Lambda[P'] \subseteq \Lambda[P]^\tau$ is surjective.

Next, we extend the argument of [38, Lem. 12.2] to prove $K_1'(\Lambda[P'])$ is a cohomologically-trivial P/P'-module with respect to the conjugation action on $\Lambda[P']$ in which P/P' acts on Λ via τ.

Before proving this we note that, if true, this fact combines directly with the argument used to prove [38, Th. 12.3] to deduce the claimed result.

Now, the argument of [38, Lem. 12.2] works verbatim in the case that P' is abelian (the only fact used in this case being that $\Lambda[P']$ is a local ring with finite residue field). To deal with the general case we let P'' be the subgroup of P' generated by central commutators in P' of order p. Then P'' is non-trivial (by [38, Lem. 6.5]) and of exponent p. We write I for the kernel of the projection map $\Lambda[P'] \to \Lambda[P'/P'']$ and $K_1'(\Lambda[P'], I)$ for the image of $K_1(\Lambda[P'], I)$ in $K_1'(\Lambda[P'])$.

Next we note that, by Lemma 2.10, the natural map $SK_1(\Lambda[P']) \to SK_1(\Lambda[P'/P''])$ is surjective and hence that there exists an exact sequence of P/P'-modules

$$1 \to K_1'(\Lambda[P'], I) \to K_1'(\Lambda[P']) \to K_1'(\Lambda[P'/P'']) \to 1. \tag{9}$$

Now, by induction on the order of commutator subgroup of P', we may assume $K_1'(\Lambda[P'/P''])$ is a cohomologically-trivial P/P'-module and so this sequence reduces us to proving that $K_1'(\Lambda[P'], I)$ is also cohomologically-trivial.

To show this one can use the same argument as in the proof of [38, Lem. 12.2]. For the convenience of the reader, we now give a sketch of this argument.

By Proposition 2.9, one knows that the image of $K_1'(\Lambda[P'], I)$ in $H_0(P', \Lambda[P'])$ under the logarithm map coincides with the image of the natural map $H_0(P', I) \to H_0(P', \Lambda[P'])$. Denoting this common image by $\overline{H}_0$ we may consider the commutative diagram

$$\begin{array}{ccccc} K_1(\Lambda[P'], I) & \longrightarrow & K_1'(\Lambda[P'], I) & \longrightarrow & 1 \\ \downarrow{\scriptstyle \log^I} & & \downarrow{\scriptstyle \log} & & \\ H_0(P', I) & \xrightarrow{\pi} & \overline{H}_0 & \longrightarrow & 0. \end{array}$$

In this diagram the map log exists, and is surjective, because the power series defining $\log(1 + x)$ converges in $H_0(P', \Lambda[P'])$ and takes values in $\overline{H}_0$. In addition, $\ker(\pi)$ is equal to the image of $H_1(P', \Lambda[P'/P''])$ in $H_0(P', I)$ and, as Λ is a flat $\mathbb{Z}$-module, the universal coefficient spectral sequence gives an isomorphism $H_1(P', \Lambda[P'/P'']) \cong H_1(P', \mathbb{Z}[P'/P''])\otimes_{\mathbb{Z}} \Lambda$. In particular, as $H_1(P', \mathbb{Z}[P'/P''])$ is finite, $\ker(\pi)$ is torsion. Since Proposition 2.8 (with P replaced by P') implies that $\ker(\log^I)$ is also torsion, one can then apply the Snake lemma to the above diagram to deduce that ker(log) is a torsion group.

However, the natural projection map $K_1'(\Lambda[P'])_{\rm tor} \to K_1'(\Lambda[P'/P''])_{\rm tor}$ is bijective (as follows, for example, directly from the explicit descriptions given in (10) and (12) below with G replaced by P' and P'/P''), and so the exact sequence (9) implies that $K_1'(\Lambda[P'], I)$ is torsion-free.

It follows that the map log is bijective and hence endows $K_1'(\Lambda[P'], I)$ with a natural action of the twisted group ring $\Lambda[P/P']^\tau$. It is then enough to note that the existence of such an action automatically implies that $K_1'(\Lambda[P'], I)$ is a cohomologically-trivial P/P'-module (see the beginning of the proof of [38, Lem. 12.2]), as required. □

2.4.2. *The proof of Proposition 2.7.* To prove claim (i) we write J_Λ and J for the Jacobson radicals of $\Lambda[G]$ and $k[G]$ respectively.

Then, since the quotient rings $\Lambda[G]/J_\Lambda$ and $k[G]/J$ are naturally isomorphic, the group $K_1(\Lambda[G]/J_\Lambda)$ is finite, of order prime to p and independent of Λ (see, for example, [38, Th. 1.16]).

We next write $K_1'(\Lambda[G], J_\Lambda)$ for the image of $K_1(\Lambda[G], J_\Lambda)$ in $K_1'(\Lambda[G])$ and claim both that $K_1'(\Lambda[G], J_\Lambda)$ is pro-p and that there is a direct sum decomposition

$$K_1'(\Lambda[G]) = K_1'(\Lambda[G], J_\Lambda) \oplus K_1'(\Lambda[G])_{[p']} \cong K_1'(\Lambda[G], J_\Lambda) \oplus K_1(k[G]/J). \tag{10}$$

In the case $\Lambda = O$ these facts are proved by Oliver in [38, Th. 2.10]. To deduce the same results for $\Lambda = O[[t]]$ one then need only fix an isomorphism of $O[[t]][G]$ with the limit

$\varprojlim_n O[\mathbb{Z}/p^n\mathbb{Z}\times G]$, where the transition maps are the natural projections, and then use the isomorphism of groups

$$K_1'(O[[t]][G]) \cong \varprojlim_n K_1'(O[\mathbb{Z}/p^n\mathbb{Z}\times G]) \tag{11}$$

that is induced by the result of Fukaya and Kato in [22, Prop. 1.5.1].

Finally, the same results for $\Lambda = O\langle t\rangle$ follow as an easy consequence of the results for O and $O[[t]]$ and the inclusions (5).

Now the decomposition (10) implies that it is enough to prove the assertion of Proposition 2.7(i) after replacing $K_1'(O[[t]][G])/K_1'(\Lambda[G])$ by $K_1'(O[[t]][G], J_{O[[t]]})/K_1'(\Lambda[G], J_\Lambda)$.

Next we note that the results of Proposition 2.11 and [16, Th. 4.7(ii)] combine with the argument that is used to prove [38, Th. 12.4] to show that $K_1'(\Lambda[G], J_\Lambda)$ is computable by restriction to p-elementary subgroups (in the terminology of Mackey functors).

Hence, after possibly extending the ring of scalars O, we may assume G is a p-group and then in this case we can prove Proposition 2.7(i) by induction on the order of commutator subgroup of G.

If G is an abelian p-group, there is nothing to prove. We therefore assume G is a non-abelian p-group and fix a central commutator z in G of order p. Then, by combining the induction hypothesis with the obvious analogue of the exact sequence (9) we are reduced to proving Proposition 2.7(i) after replacing $K_1'(O[[t]][G])/K_1'(\Lambda[G])$ by the quotient

$$K_1'(O[[t]][G], I_z)/K_1'(\Lambda[G], (1-z)\Lambda[G])$$

with $I_z := (1-z)O[[t]][G]$.

Now, as z is a commutator, the argument of Proposition 2.9 shows that the image of $K_1'(O[[t]][G], I_z)$ under the logarithm map lies in $H_0(G, O[[t]][G])$ and coincides with the image of $H_0(G, I_z)$ in $H_0(G, O[[t]][G])$. Hence, since the image of $\langle z\rangle$ in $K_1'(O[[t]][G], I_z)$ is trivial, the exact sequence in Proposition 2.8 implies that $K_1'(O[[t]][G], I_z)$ is isomorphic to the image of $H_0(G, I_z)$ in $H_0(G, O[[t]][G])$.

In this way, the proof of Proposition 2.7(i) is reduced to showing that the quotient of the respective images in $H_0(G, O[[t]][G])$ of $H_0(G, I_z)$ and $H_0(G, (1-z)\Lambda[G])$ is torsion-free. Then, since $H_0(G, O[[t]][G])$ is a free $O[[t]]$-module on the conjugacy classes of G, this is easily deduced from the fact that the quotient of $O[[t]]$ by Λ is torsion-free.

Turning to the proof of Proposition 2.7(ii), we note that the decomposition (10) immediately implies that $K_1'(O[G])_{[p']} = K_1'(\Lambda[G])_{[p']}$ and so it suffices to show that $K_1'(O[G])_{[p]} = K_1'(\Lambda[G])_{[p]}$.

To do this we fix a set $\{g_1, \dots, g_k\}$ of representatives of the set of L-conjugacy classes of elements of G that are of order prime to p. We write Z_i for the centralizer of g_i in G, n_i for the order of g_i, μ_{n_i} for the group of n_i^{th} roots of unity in $\mathbb{Q}_p^c$ and set

$$N_i := \{x \in G : xg_ix^{-1} = g_i^a \text{ for some } a \in \mathrm{Gal}(L(\mu_{n_i})/L)\}.$$

Then, in view of the inclusions (5), it is enough to show that there is a natural group isomorphism

$$K_1'(\Lambda[G])_{[p]} \cong (\mu(L)_{[p]})^{\oplus k} \oplus \bigoplus_{i=1}^{i=k} H^0(N_i/Z_i, Z_i^{\text{ab}})_{[p]} \tag{12}$$

for both $\Lambda = O$ and $\Lambda = O[[t]]$ (and hence also for $\Lambda = O\langle t\rangle$). But if $\Lambda = O$, this is proved by Oliver in [38, Th. 12.5(ii)] and one then obtains the same isomorphism in the case $\Lambda = O[[t]]$ by combining (11) with the given isomorphisms for the rings $O[\mathbb{Z}/p^n\mathbb{Z}\times G]$ for every natural number n.

This completes the proof of Proposition 2.7.

2.4.3. *Torsion and the condition ($F_{\Lambda[G]}$).* The results that are proved in this section will show that the group $\theta_{\Lambda[G]}(K_1'(\Lambda[G])_{\rm tor})$ is characterized solely by the condition ($\mathrm{F}_{\Lambda[G]}$) in Theorem 2.1.

We first consider torsion elements of order prime to p.

Proposition 2.12. *$\theta_{\Lambda[G]}$ restricts to give an isomorphism between $K_1'(\Lambda[G])_{[p']}$ and the subgroup of $\prod_{U\in S(G)}(\Lambda[U^{\rm ab}]^\times)_{[p']}$ comprising elements that satisfy the condition ($F_{\Lambda[G]}$).*

Proof. In view of Proposition 2.7(ii) it is enough to prove the claim in the case that $\Lambda = O$. To do this we write $\mathcal{B}_k(G)$ for the set of k-irreducible modular (or 'Brauer') characters of G and for each ρ in $\mathcal{B}_k(G)$ we denote the smallest field extension of k over which ρ can be realised by $k(\rho)$.

Then there are natural isomorphisms

$$K_1'(O[G])_{[p']} \cong K_1(O[G]/J_O) \cong \prod_{\rho\in\mathcal{B}_k(G)} k(\rho),$$

where the first is described by Oliver in [38, Th. 2.10] and the second is proved in [17, 18.1]. With respect to these isomorphisms, the restriction of $\theta_{O[G]}$ to $K_1'(O[G])_{[p']}$ is identified with the homomorphism

$$\prod_{\rho\in\mathcal{B}_k(G)} k(\rho) \to \prod_{U\in S(G)}\prod_{\chi\in\mathcal{B}_k(U^{\rm ab})} k(\chi)$$

that sends each element $(a_\rho)_\rho$ to $(\prod_\rho a_\rho^{n_\rho(U,\chi)})_{U,\chi}$ where the integers $n_\rho(U,\chi)$ are defined via the equality of characters

$$(i_U^G)_*\circ(i_{U^{\rm ab}}^U)^*(\chi) = \sum_{\rho\in\mathcal{B}_k(G)} n_\rho(U,\chi)\cdot\rho.$$

Given this explicit description of the restriction of $\theta_{O[G]}$, the claimed result follows directly from [17, Exer. 18.13]. □

Proposition 2.13. *$\theta_{\Lambda[G]}$ restricts to give an isomorphism between $K_1'(\Lambda[G])_{[p]}$ and the subgroup of $\prod_{U\in S(G)}(\Lambda[U^{\rm ab}]^\times)_{[p]}$ comprising elements that satisfy the condition ($F_{\Lambda[G]}$).*

Proof. In view of Propositions 2.5, 2.6 and 2.7(ii) it suffices to show that any given element (x_U) of $\prod_{U\in S(G)}(O[U^{\rm ab}]^\times)_{[p]}$ that satisfies ($\mathrm{F}_{O[G]}$) belongs to $\theta_{O[G]}(K_1'(O[G])_{[p]})$.

To do this we write $E_p(G)$ for the set of p-elementary subgroups of G and note that for any U in $E_p(G)$ the isomorphism (12) with $\Lambda = O$ and $G = U$ implies that the natural projection map $K_1'(O[U])_{[p]} \to K_1'(O[U^{\rm ab}])_{[p]} = (O[U^{\rm ab}]^\times)_{[p]}$ is bijective.

Recalling that $K_1'(O[G], J_O)$ is computable by restriction to p-elementary subgroups (see the proof of Proposition 2.7(i)), it is therefore enough to show that if U and V are any two subgroups in $E_p(G)$ then the given element (x_U) has both of the following properties:

(a) If $U = gVg^{-1}$ for some g in G, then $c_g(x_U) = x_V$ where c_g denotes the map induced by conjugation by g.
(b) If $V \subseteq U$, then $(i_V^U)^*(x_U) = x_V$.

This is in turn true because both of these properties can be deduced as a consequence of the condition ($\mathrm{F}_{O[G]}$). For brevity, we shall only derive (b) from ($\mathrm{F}_{O[G]}$), leaving the analogous (but easier) derivation of (a) to the reader.

Now, to prove (b) it suffices to show that $D_{O[V]}(x_V)(\chi) = D_{O[V]}((i_V^U)^*(x_U))(\chi)$ for every χ in $R(V)$. But for any such χ one has $(i_V^G)_*(\chi) - (i_U^G)_*(i_V^U)_*(\chi) = 0$ and hence, by the assumed validity of ($\mathrm{F}_{O[G]}$), also

$$D_{O[V]}(x_V)(\chi) = D_{O[U]}(x_U)((i_V^U)_*(\chi)) = D_{O[V]}((i_V^U)^*(x_U))(\chi),$$

as required. □

2.4.4. *Torsion and the elements η_U.* In this section we prove the following result.

Proposition 2.14. *For any element (x_U) of $\prod_{U\in S(G)}\Lambda[U^{\mathrm{ab}}]^\times$ that satisfies the condition ($F_{\Lambda[G]}$), the following properties are equivalent.*

(i) *$\eta_U \cdot x_U$ is torsion for every cyclic subgroup U of G.*
(ii) *x_U is torsion for every cyclic subgroup U of G.*
(iii) *x_U is torsion for every subgroup U of G.*

Proof. At the outset we note that the implication from (iii) to (i) is obvious and that an element y of $K_1'(\Lambda[U])$ is torsion if and only if the element $D_{\Lambda[U]}(y)(\chi)$ of $\Lambda_{\mathbb{Q}_p^c}^\times$ is torsion for every χ in $R(U)$.

To prove the implication from (i) to (ii) we fix U in $C(G)$ and for χ in $R(U)$ and V in $C(U)$ we set $\chi_V = (i_V^U)^*(\chi)$. Then the equality (8) implies that for each such χ the element

$$D_{\Lambda[U]}(x_U)(\chi)^{|U|} = D_{\Lambda[U]}(x_U)(|U|\cdot\chi)$$

is equal to

$$\begin{aligned} D_{\Lambda[U]}(x_U)(\textstyle\sum_{V\in C(U)}(i_V^U)_*(\eta_V\cdot\chi_V)) &= \prod_{V\in C(U)} D_{\Lambda[U]}(x_U)((i_V^U)_*(\eta_V\cdot\chi_V)) \\ &= \prod_{V\in C(U)} D_{\Lambda[V]}((i_V^U)^*(x_U))(\eta_V\cdot\chi_V) \\ &= \prod_{V\in C(U)} D_{\Lambda[V]}(x_V)(\eta_V\cdot\chi_V) \\ &= \prod_{V\in C(U)} D_{\Lambda[V]}(\eta_V\cdot x_V)(\chi_V) \end{aligned}$$

where the penultimate equality is a consequence of Lemma 2.15 below. Thus, since claim (i) implies that the last displayed product is a torsion element, it follows that $D_{\Lambda[U]}(x_U)(\chi)$ is torsion for all χ in $R(U)$ and hence that (ii) is valid.

To show the implication from (ii) to (iii) we need to show that x_U is a torsion element for every U in $S(G)$. Arguing by contradiction we take a subgroup U of least order such that x_U is not torsion. Then U cannot be cyclic but $U^{\rm ab}$ can be cyclic and so we have to consider two different cases.

We assume first that $U^{\rm ab}$ is cyclic. Then there exists a V in $C(U)$ for which the natural map $V \to U^{\rm ab}$ is surjective and so induces an isomorphism between $U^{\rm ab}$ and the quotient $V' := V/(V\cap [U,U])$. In particular, by Lemma 2.15 below, one has $x_U = (i_{V'}^{V^{\rm ab}})_*(x_V)$ and so, since x_V is (by assumption) torsion the element x_U is torsion, as required.

Next we assume $U^{\rm ab}$ is not cyclic. Then for every V in $C(U^{\rm ab})$ the full pre-image $\tilde V$ of V in U is a proper subgroup of U and so $x_{\tilde V}$ is torsion. The image x_V of $x_{\tilde V}$ under the natural projection map $\Lambda[\tilde V^{\rm ab}]^\times \to \Lambda[V]^\times$ is therefore also torsion and, by combining Lemma 2.15 below with the construction of x_V, one knows that $(i_V^{U^{\rm ab}})^*(x_U)$ is equal to x_V.

Hence, if we set $u := |U^{\rm ab}|$ and for any χ in $R(U^{\rm ab})$ also $\chi_V := (i_V^{U^{\rm ab}})^*(\chi)$, then one has

$$\begin{aligned} D_{\Lambda[U]}(x_U)(\chi)^u &= D_{\Lambda[U]}(x_U)(u\cdot\chi)\\ &= D_{\Lambda[U]}(x_U)\Big(\sum_{V\in C(U^{\rm ab})} (i_V^{U^{\rm ab}})_*(\eta_V\cdot\chi_V)\Big)\\ &= \prod_{V\in C(U^{\rm ab})} D_{\Lambda[V]}((i_V^{U^{\rm ab}})^*(x_U))(\eta_V\cdot\chi_V)\\ &= \prod_{V\in C(U^{\rm ab})} D_{\Lambda[V]}(x_V)(\eta_V\cdot\chi_V), \end{aligned}$$

where the second equality follows from (8) with W replaced by $U^{\rm ab}$. In particular, since each element x_V that occurs in the last product is torsion, this proves claim (iii). □

Lemma 2.15. *Let (x_U) be an element of $\prod_{U\in S(G)}\Lambda[U^{\rm ab}]^\times$ that satisfies condition $(F_{\Lambda[G]})$. Fix subgroups U and V of G with $V\subseteq U$ and set $V' := V/(V\cap[U,U])$. Then one has*

$$(i_{V'}^{U^{\rm ab}})^*(x_U^{[[U,U]V:V]}) = (i_{V'}^{V^{\rm ab}})_*(x_V).$$

Proof. Set $d := [[U,U]V : V]$. Then it suffices to prove that for every χ in $R(V')$ one has

$$D_{\Lambda[V']}((i_{V'}^{U^{\rm ab}})^*(x_U^d))(\chi) = D_{\Lambda[V']}((i_{V'}^{V^{\rm ab}})_*(x_V))(\chi).$$

In addition, since

$$D_{\Lambda[V']}((i_{V'}^{U^{\rm ab}})^*(x_U^d))(\chi) = D_{\Lambda[U]}(x_U)((i_{V'}^{U^{\rm ab}})_*(d\cdot\chi))$$

and

$$D_{\Lambda[V']}((i_{V'}^{V^{\rm ab}})_*(x_V))(\chi) = D_{\Lambda[V]}(x_V)((i_{V'}^{V^{\rm ab}})^*(\chi)),$$

the required equality follows directly by applying $(\mathrm{F}_{\Lambda[G]})$ to the character relation

$$(i_U^G)_*(i_{U^{\rm ab}}^U)^*(i_{V'}^{U^{\rm ab}})_*(d\cdot\chi) - (i_V^G)_*(i_{V^{\rm ab}}^V)^*(i_{V'}^{V^{\rm ab}})^*(\chi) = 0.$$

□

2.5. **The proofs of Theorem 2.1 and Corollary 2.3.**

2.5.1. The results of Propositions 2.5 and 2.6 reduce the proof of Theorem 2.1 to showing that any element $(x_U)_U$ of $\prod_{U\in S(G)}\Lambda[U^{\rm ab}]^\times$ that satisfies both of the conditions $({\rm F}_{\Lambda[G]})$ and $({\rm C}_{\Lambda[G]})$ must belong to ${\rm im}(\theta_{\Lambda[G]})$.

To prove this we note, firstly, that since such an element (x_U) satisfies condition $({\rm C}_{\Lambda[G]})$ we can choose an element x in $K_1'(\Lambda[G])$ with $\prod_{U\in C(G)}(i_U^G)_*(\eta_U\cdot x_U)=x^{|G|}$.

Then the standard 'double coset formula' for Mackey functors implies that, for each V in $C(G)$, the element

$$\theta_V(x^{|G|})=(i_V^G)^*(\prod_{U\in C(G)}(i_U^G)_*(\eta_U\cdot x_U))=\prod_{U\in C(G)}(i_V^G)^*(i_U^G)_*(\eta_U\cdot x_U)$$

can be computed as

$$\begin{aligned}&\prod_{U\in C(G)}\prod_{g\in V\backslash G/U}\big((i^V_{V\cap gUg^{-1}})_*\circ c_g\circ(i^U_{g^{-1}Vg\cap U})^*\big)(\eta_U\cdot x_U)\\ =&\prod_{g\in V\backslash G}\prod_{\substack{U\in C(G)\\ gUg^{-1}\subseteq V}}(i^V_{gUg^{-1}})_*(\eta_{gUg^{-1}}\cdot x_{gUg^{-1}})\\ =&\prod_{g\in V\backslash G}\prod_{U\subseteq V}(i_U^V)_*(\eta_U\cdot x_U)\\ =&\big(\sum_{U\subseteq V}((i_U^V)_*(\eta_U))(x_V)\big)^{[G:V]}\\ =&(|V|\cdot x_V)^{[G:V]}\\ =&x_V^{|G|}.\end{aligned}$$

Here the first equality is valid since every subgroup $V\cap gUg^{-1}$ is equal to $gU'g^{-1}$ for some $U'\in C(G)$ and also Lemma 2.15 implies $(i^U_{g^{-1}Vg\cap U})^*(\eta_U\cdot x_U)=\eta_{g^{-1}Vg\cap U}\cdot x_{g^{-1}Vg\cap U}$ for all g and U. The second equality is clear and the third follows from the Frobenius axiom and the fact that Lemma 2.15 implies $(i_U^V)^*(x_V)=x_U$ for all $U\subseteq V$. Finally, the fourth equality follows from the equality (8) (with W replaced by each V) and the last is clear.

The above computation implies that, after replacing (x_U) by $\theta(x)^{-1}(x_U)$ if necessary, we can assume that (x_U) satisfies both $({\rm F}_{\Lambda[G]})$ and $({\rm C}_{\Lambda[G]})$ and is also such that x_U is torsion for all U in $C(G)$. Then, in this case, Proposition 2.14 implies that x_U is torsion for all subgroups U of G.

We can therefore now apply Propositions 2.12 and 2.13 to deduce that (x_U) belongs to ${\rm im}(\theta_{\Lambda[G]})$, as required to complete the proof of Theorem 2.1.

2.5.2. To prove Corollary 2.3 it is clearly enough to show that any element (x_U) of the intersection ${\rm im}(\theta_{O[[t]][G]})\cap\prod_{U\in S(G)}O\langle t\rangle[U^{\rm ab}]^\times$ belongs to ${\rm im}(\theta_{O\langle t\rangle[G]})$.

But, fixing such an element (x_U), and then an element x of $K_1'(O[[t]][G])$ with $\theta_{O[[t]][G]}(x)$ $=(x_U)$, the argument of Proposition 2.6 implies

$$x^{|G|}=\prod_{U\in C(G)}(i_U^G)_*(\eta_U\cdot x_U)$$

and hence that $x^{|G|}$ belongs to $K_1'(O\langle t\rangle[G])$.

Thus, since $\theta_{O[[t]][U]}(x) = x_U$ belongs to $O\langle t\rangle[U^{\rm ab}]^\times$ for every U in $S(G)$, Proposition 2.7(i) implies that x belongs to $K_1'(O\langle t\rangle[G])$ and hence that

$$(x_U) = \theta_{O[[t]][G]}(x) = \theta_{O\langle t\rangle[G]}(x)$$

belongs to $\mathrm{im}(\theta_{O\langle t\rangle[G]})$, as required.

This completes the proof of Corollary 2.3.

3. Euler characteristics, Bockstein homomorphisms and semisimplicity

The aim of this section is to recall several useful facts from both algebraic K-theory and homological algebra.

To do so we fix a Dedekind domain R with field of fractions F, an R-order $\mathcal{A}$ in a finite dimensional separable F-algebra A and a field extension E of F. We write A_E for the semisimple E-algebra $E\otimes_F A$.

3.1. **Euler characteristics.**

3.1.1. We use the relative algebraic K_0-group $K_0(\mathcal{A}, A_E)$ of the ring inclusion $\mathcal{A}\subset A_E$, as described explicitly in terms of generators and relations by Swan in [43, p. 215].

We recall, in particular, that for any extension field E' of E there exists an exact commutative diagram

$$\begin{array}{ccccccc} K_1(\mathcal{A}) & \longrightarrow & K_1(A_{E'}) & \xrightarrow{\partial_{\mathcal{A},E'}} & K_0(\mathcal{A},A_{E'}) & \xrightarrow{\partial'_{\mathcal{A},E'}} & K_0(\mathcal{A}) \\ \| & & \iota\uparrow & & \iota'\uparrow & & \| \\ K_1(\mathcal{A}) & \longrightarrow & K_1(A_E) & \xrightarrow{\partial_{\mathcal{A},E}} & K_0(\mathcal{A},A_E) & \xrightarrow{\partial'_{\mathcal{A},E}} & K_0(\mathcal{A}) \end{array} \tag{13}$$

in which the upper and lower rows are the respective long exact sequences in relative K-theory of the inclusions $\mathcal{A}\subset A_E$ and $\mathcal{A}\subset A_{E'}$ and both of the vertical arrows are injective and induced by the inclusion $A_E\subseteq A_{E'}$. (For more details see [43, Th. 15.5].) In the sequel we will usually identify the groups $K_1(A_E)$ and $K_0(\mathcal{A},A_E)$ with their respective images under the maps ι and ι'.

If $R=\mathbb{Z}$ and for each prime ℓ we set $\mathcal{A}_\ell := \mathbb{Z}_\ell\otimes_{\mathbb{Z}}\mathcal{A}$ and $A_\ell := \mathbb{Q}_\ell\otimes_{\mathbb{Q}} A$, then we can regard each group $K_0(\mathcal{A}_\ell, A_\ell)$ as a subgroup of $K_0(\mathcal{A},A)$ by means of the canonical composite homomorphism

$$\bigoplus_\ell K_0(\mathcal{A}_\ell, A_\ell)\cong K_0(\mathcal{A},A)\subset K_0(\mathcal{A},A_{\mathbb{R}}), \tag{14}$$

where ℓ runs over all primes, the isomorphism is as described in the discussion following [17, (49.12)] and the inclusion is induced by the relevant case of ι'. For each element x of $K_0(\mathcal{A},A)$ we write $(x_\ell)_\ell$ for its image in $\bigoplus_\ell K_0(\mathcal{A}_\ell,A_\ell)$ under the isomorphism in (14).

If R is the valuation ring of a finite extension of $\mathbb{Q}_\ell$ for some prime ℓ, then there exists a composite homomorphism

$$\delta_{\mathcal{A}}: \zeta(A)^\times\to K_1(A)\xrightarrow{\partial_{\mathcal{A},A}} K_0(\mathcal{A},A) \tag{15}$$

in which the first map is the inverse of the (bijective) homomorphism $K_1(A) \to \zeta(A)^\times$ that is induced by taking reduced norms over A.

Finally, we note that the argument of [6, §4.2, Lem. 9] implies the existence of a canonical 'extended boundary homomorphism' of relative K-theory δ_G that, for every prime ℓ, lies in a commutative diagram of the form

$$\begin{array}{ccc} \zeta(\mathbb{Q}[G])^\times & \xrightarrow{\delta_G} & K_0(\mathbb{Z}[G], \mathbb{Q}[G]) \\ {\scriptstyle \iota_\ell}\downarrow & & \downarrow{\scriptstyle \pi_\ell} \\ \zeta(\mathbb{Q}_\ell[G])^\times & \xrightarrow{\delta_{\mathbb{Z}_\ell[G]}} & K_0(\mathbb{Z}_\ell[G], \mathbb{Q}_\ell[G]). \end{array} \tag{16}$$

in which ι_ℓ denotes the natural inclusion and π_ℓ the projection induced by (14).

3.1.2. We also find it convenient to use a description of $K_0(\mathcal{A}, A_E)$ in terms of the formalism of '(non-abelian) determinant categories' introduced by Fukaya and Kato in [22, §1]. We recall, in particular, that any pair comprising an object C of $D^{\rm perf}(\mathcal{A})$ and a morphism h of (non-abelian) determinants $\mathrm{Det}_{A_E}(E \otimes_R C) \to \mathrm{Det}_{A_E}(0)$ gives rise to a canonical element of $K_0(\mathcal{A}, A_E)$ that we shall denote by $\chi_{\mathcal{A}}(C, h)$. (For precise details of the relation between this construction and the localized K_1-groups of Fukaya and Kato, see [7, §4.1.2].)

For example, if C is an object of $D^{\rm perf}(\mathcal{A})$, then any exact sequence of A_E-modules of the form

$$\epsilon:\ 0 \to \cdots \to E \otimes_R H^i(C) \to E \otimes_R H^{i+1}(C) \to E \otimes_R H^{i+2}(C) \to \cdots \to 0$$

induces a canonical morphism $\mathrm{Det}_{A_E}(\epsilon) : \mathrm{Det}_{A_E}(E \otimes_R C) \to \mathrm{Det}_{A_E}(0)$ and hence gives rise to a canonical 'Euler characteristic' element

$$\chi_{\mathcal{A}}(C, \epsilon) := (\mathrm{Det}_{\mathcal{A}}(C), \mathrm{Det}_{A_E}(\epsilon))$$

in $K_0(\mathcal{A}, A_E)$. In particular, if an object C of $D^{\rm perf}(\mathcal{A})$ is such that $F \otimes_R C$ (or, equivalently, $E \otimes_R C$) is acyclic, then we can take ϵ as the zero sequence and so obtain a canonical element $\chi_{\mathcal{A}}(C, 0)$ of $K_0(\mathcal{A}, A) \subseteq K_0(\mathcal{A}, A_E)$.

There are two standard properties of such elements that are established by Fukaya and Kato in loc. cit. and will be frequently used (without explicit comment) in the sequel.

Firstly, if

$$C_1 \to C_2 \to C_3 \to C_1[1]$$

is any exact triangle in $D^{\rm perf}(\mathcal{A})$ such that the complex $F \otimes_R C_3$ is acyclic, then any morphism $h : \mathrm{Det}_{A_E}(E \otimes_R C_1) \to \mathrm{Det}_{A_E}(0)$ combines with the given triangle to induce a morphism $h' : \mathrm{Det}_{A_E}(E \otimes_R C_2) \to \mathrm{Det}_{A_E}(0)$ for which one has

$$\chi_{\mathcal{A}}(C_2, h') = \chi_{\mathcal{A}}(C_1, h) + \chi_{\mathcal{A}}(C_3, 0) \tag{17}$$

in $K_0(\mathcal{A}, A_E)$.

Secondly, if h and h' are any two morphisms $\mathrm{Det}_{A_E}(E \otimes_R C) \to \mathrm{Det}_{A_E}(0)$, then one has

$$\chi_{\mathcal{A}}(C, h') = \chi_{\mathcal{A}}(C, h) + \partial_{\mathcal{A},E}(h' \circ h^{-1})$$

in $K_0(\mathcal{A}, A_E)$, where the composite $h' \circ h^{-1} : \mathrm{Det}_{A_E}(0) \to \mathrm{Det}_{A_E}(0)$ is regarded as an element of $K_1(A_E)$ in the natural way.

Remark 3.1. For convenience, we shall often abbreviate the notation $\chi_{\mathbb{Z}[G]}(C,h)$ to $\chi_G(C,h)$ and also, when E is clear from context, write ∂_G and ∂'_G in place of $\partial_{\mathbb{Z}[G],E}$ and $\partial'_{\mathbb{Z}[G],E}$.

3.2. **Bockstein homomorphisms and semisimplicity.** For each $\mathcal{A}$-module M we write $F\cdot M$ for the asociated A-module $F\otimes_R M$.

3.2.1. We recall that an endomorphism ψ of a finitely generated $\mathcal{A}$-module M is said to be 'semi-simple at 0' if the natural composite homomorphism

$$F\cdot\ker(\psi)\xrightarrow{\subseteq}F\cdot M\twoheadrightarrow F\cdot\mathrm{coker}(\psi) \tag{18}$$

is bijective.

This condition is satisfied if and only if there exists an $A[\psi]$-equivariant direct complement to $F\cdot\ker(\psi)$ in $F\cdot M$. In particular, in such a case ψ induces an automorphism $\psi^\diamond$ of any such direct complement and one checks easily that $\mathrm{Nrd}_A(\psi^\diamond)$ is independent of the choice of this complement.

3.2.2. We now suppose to be given an exact triangle in $D^{\mathrm{perf}}(\mathcal{A})$ of the form

$$C\xrightarrow{\theta}C\xrightarrow{\theta_1}C_\theta\xrightarrow{\theta_2}C[1]. \tag{19}$$

This triangle gives rise in each degree i to a composite 'Bockstein homomorphism'

$$\beta^i_\theta: H^i(C_\theta)\xrightarrow{H^i(\theta_2)}H^{i+1}(C)\xrightarrow{H^{i+1}(\theta_1)}H^{i+1}(C_\theta)$$

and hence to an associated sequence of $\mathcal{A}$-modules

$$\cdots\xrightarrow{\beta^{i-1}_\theta}H^i(C_\theta)\xrightarrow{\beta^i_\theta}H^{i+1}(C_\theta)\xrightarrow{\beta^{i+1}_\theta}\cdots. \tag{20}$$

The following technical result will play an important role in the sequel.

Proposition 3.2.

(i) *The endomorphisms $H^i(\theta)$ of the $\mathcal{A}$-modules $H^i(C)$ are semisimple at zero in all degrees i if and only if the sequence ϵ_θ of A-modules that is induced by (20) is exact.*

(ii) *If the sequence ϵ_θ is exact, then the Euler characteristic $\chi_{\mathcal{A}}(C_\theta,\epsilon_\theta)$ is well-defined and in $K_0(\mathcal{A},A)$ one has*

$$\chi_{\mathcal{A}}(C_\theta,\epsilon_\theta)=\partial_{\mathcal{A},A}\big(\textstyle\prod_{i\in\mathbb{Z}}(H^i(\theta)^\diamond_F)^{(-1)^i}\big),$$

where we identify each automorphism $H^i(\theta)^\diamond_F$ with the associated element of $K_1(A)$.

Proof. Claim (i) is both well-known and straightforward to verify directly. Claim (ii) then follows as a special case of [7, Prop. 5.1] (or, alternatively, see [2, Prop. 3.1]). □

Remark 3.3. In each degree i one has $\mathrm{im}(\beta^{i-1}_\theta)\subseteq\ker(\beta^i_\theta)$ and so the sequence (20) is a complex. In particular, the key 'semisimplicity' condition in Proposition 3.2 is equivalent to requiring that the cohomology groups of this complex are finite in all degrees.

4. Zeta functions and Euler characteristics for smooth flat sheaves.

After making certain preliminary observations of a general nature, we shall in this section state, for an arbitrary prime ℓ, a K-theoretical leading term formula for the Zeta functions of ℓ-adic étale sheaves relative to finite Galois covers of schemes over $\mathbb{F}_q$ (for details see Theorem 4.3). In the remainder of the section we will then reduce the proof of this result to the special case that $\ell = p$.

4.1. **General observations.**

4.1.1. Let $s_X : X \to \mathrm{Spec}(\mathbb{F}_q)$ be a separated morphism of finite type and write $\mathbf{FEt}/X$ for the category of X-schemes that are finite and étale over X.

Then for any geometric point $\overline{x}$ of X the functor that takes each scheme Y to the set $F_{\overline{x}}(Y) := \mathrm{Hom}_X(\overline{x}, Y)$ of geometric points of Y that lie over $\overline{x}$ gives an equivalence of categories between $\mathbf{FEt}/X$ and the category of finite sets upon which $\pi_1(X, \overline{x})$ acts continuously (on the left).

For a morphism $f : Y \to X$ in $\mathbf{FEt}/X$, the finite group $\mathrm{Aut}_X(Y)$ acts naturally on $F_{\overline{x}}(Y)$ (on the right) and if Y is connected, then it is said to be 'Galois over X of group $G := \mathrm{Aut}_X(Y)$' if for any ξ in $F_{\overline{x}}(Y)$ the map $\mathrm{Aut}_X(Y) \to F_{\overline{x}}(Y)$ that sends each σ to $\sigma \circ \xi$ is bijective.

If now G is any finite topological quotient of the fundamental group $\pi_1(X, \overline{x})$, then there exists a canonical connected scheme in $\mathbf{FEt}/X$ that is Galois of group G. We denote this scheme by $(f : Y \to X, G)$, or more simply by $f : Y \to X$, and refer to it as the 'covering of X of group G'.

In this case there exists for each subgroup H of G a diagram in $\mathbf{FEt}/X$ of the form

$$\begin{array}{ccc} Y & \xrightarrow{\;f^H\;} & Y^H \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f_H} & \\ & X & \end{array} \tag{21}$$

in which the morphism f^H is Galois over Y^H of group H and, if H is normal in G, then f_H is Galois over X of group G/H.

4.1.2. We now fix a prime number ℓ, a finite extension Q of $\mathbb{Q}_\ell$ with valuation ring O and an étale sheaf of O-modules $\mathcal{L}$ on X.

Then for any morphism $h : Y \to X$ in $\mathbf{FEt}/X$ the sheaf $h_*h^*\mathcal{L}$ can be explicitly described as the sheaf of O-modules that is associated to the pre-sheaf $U \mapsto \bigoplus_{\mathrm{Hom}_X(U,Y)} \mathcal{L}(U)$ where the transition morphisms $\bigoplus_{\mathrm{Hom}_X(V,Y)} \mathcal{L}(V) \to \bigoplus_{\mathrm{Hom}_X(U,Y)} \mathcal{L}(U)$ for $\alpha : U \to V$ map $(x_\psi)_\psi$ to $\sum_{\psi\circ\alpha} \mathcal{L}(\alpha)(x_\psi)$.

If h is Galois of group G, then, by a slight abuse of notation, we shall set $\mathcal{L}_G := h_*h^*\mathcal{L}$. In this case the natural right action of G on $\mathrm{Hom}_X(U, Y)$ gives a left action on $h_*h^*\mathcal{L}$ by permuting the components and for the stalk at $\overline{x}$ there is therefore an isomorphism of $O[G]$-modules

$$(\mathcal{L}_G)_{\overline{x}} \cong \bigoplus_{F_{\overline{x}}(Y)} \mathcal{F}_{\overline{x}} \cong O[G] \otimes_O \mathcal{L}_{\overline{x}}. \tag{22}$$

Here the second map results from the fact that (since h is Galois) any choice of an element ξ of $F_{\overline{x}}(Y)$ induces an isomorphism $\sigma \mapsto \sigma \circ \xi$ of G-sets $G \cong F_{\overline{x}}(Y)$.

4.1.3. Since s_X is separated we may fix a factorisation $s_X = s' \circ j$ with $j : X \to X'$ an open immersion and $s' : X' \to \mathrm{Spec}(\mathbb{F}_q)$ a proper morphism. The direct image with proper support $Rs_{X!}\mathcal{L}_G$ of $\mathcal{L}_G$ can then be defined as the complex of $O[G]$-sheaves $s'_*G^\bullet_{X'}(j_!\mathcal{L}_G)$, where $G^\bullet_{X'}(\mathcal{K})$ denotes the Godement resolution of an étale sheaf $\mathcal{K}$ on X'.

One then defines the cohomology with compact support of $\mathcal{L}_G$ and the higher direct image with proper support of $\mathcal{L}_G$ by respectively setting

$$R\Gamma_c(X_{\text{ét}}, \mathcal{L}_G) := R\Gamma(X'_{\text{ét}}, j_!\mathcal{L}_G)$$

and

$$R\Gamma(\mathbb{F}_q, Rs_{X,!}\mathcal{L}_G) := R\Gamma(\mathrm{Spec}(\mathbb{F}_q)_{\text{ét}}, Rs_{X,!}\mathcal{L}_G).$$

We recall that the arguments of Deligne in [18, Arcata, IV, §5] show that these complexes are each independent, up to natural isomorphism in $D(O[G])$, of the chosen compactification $s_X = s' \circ j$.

In the following result we write s_c for the natural morphism $\mathrm{Spec}(\mathbb{F}_q^c) \to \mathrm{Spec}(\mathbb{F}_q)$. We also recall (from the Introduction) that ϕ denotes the geometric Frobenius automorphism in $\mathrm{Gal}(\mathbb{F}_q^c/\mathbb{F}_q)$ and we use the fact that $\mathrm{Gal}(\mathbb{F}_q^c/\mathbb{F}_q)$ acts naturally on the complex $R\Gamma(\mathbb{F}_q^c, s_c^*Rs_{X,!}\mathcal{L}_G)$.

Proposition 4.1. *Let $f : Y \to X$ be a Galois cover of group G. Let O be a finite extension of $\mathbb{Z}_\ell$ and $\mathcal{L}$ a flat and smooth O-sheaf on X. Then the following claims are valid.*

(i) *$R\Gamma(\mathbb{F}_q, Rs_{X,!}\mathcal{L}_G)$ is naturally isomorphic to $R\Gamma_c(X_{\text{ét}}, \mathcal{L}_G)$.*

(ii) *$R\Gamma(\mathbb{F}_q, Rs_{X,!}\mathcal{L}_G)$ and $R\Gamma(\mathbb{F}_q^c, s_c^*Rs_{X,!}\mathcal{L}_G)$ belong to $D^{\mathrm{perf}}(O[G])$.*

(iii) *There is a canonical exact triangle in $D^{\mathrm{perf}}(O[G])$ of the form*

$$R\Gamma(\mathbb{F}_q^c, s_c^*Rs_{X,!}\mathcal{L}_G) \xrightarrow{1-\phi} R\Gamma(\mathbb{F}_q^c, s_c^*Rs_{X,!}\mathcal{L}_G) \to R\Gamma_c(X_{\text{ét}}, \mathcal{L}_G)[1] \to .$$

(iv) *The Euler characteristic of $R\Gamma_c(X_{\text{ét}}, \mathcal{L}_G)$ in $K_0(O[G])$ vanishes.*

Proof. Claim (i) follows directly from the definitions of $R\Gamma(\mathbb{F}_q, Rs_{X,!}\mathcal{L}_G)$ and $R\Gamma_c(X_{\text{ét}}, \mathcal{L}_G)$ via the chosen compactification $s_X = s' \circ j$ and the fact there are natural isomorphisms of complexes

$$R\Gamma(\mathbb{F}_q, s'_*G^\bullet_{X'}(j_!\mathcal{F})) \cong R\Gamma(\mathbb{F}_q, Rs'_*(j_!\mathcal{F})) \cong R\Gamma(X'_{\text{ét}}, j_!\mathcal{F}).$$

Claim (ii) follows directly from the result of [18, p. 95, Th. 4.9] and the fact that the isomorphism (22) implies each stalk of $\mathcal{L}_G$ is a finitely generated projective $O[G]$-module.

Claim (iii) is a well known consequence of the Hochschild-Serre spectral sequence and the isomorphism in claim (i).

Finally, claim (iv) follows immediately upon taking Euler characteristics (over $O[G]$) of the complexes in the exact triangle in claim (iii). □

Remark 4.2. For later use, it is convenient to discuss the exact triangle in Proposition 4.1(iii) in terms of the formalism described in §3.2.

(i) We consider the sequence of $Q[G]$-modules

$$\epsilon^{\mathrm{Bock}}_{\mathcal{L},G,\phi} : \quad \cdots \xrightarrow{Q\otimes_O\beta^{i-1}_{\mathcal{L},G,\phi}} H^i_c(X_{\text{ét}}, Q\otimes_O \mathcal{L}_G) \xrightarrow{Q\otimes_O\beta^{i}_{\mathcal{L},G,\phi}} H^{i+1}_c(X_{\text{ét}}, Q\otimes_O \mathcal{L}_G) \xrightarrow{Q\otimes_O\beta^{i+1}_{\mathcal{L},G,\phi}} \cdots,$$

where $\beta^i_{\mathcal{L},G,\phi}$ denotes the Bockstein homomorphism $H^i_c(X_{\text{ét}},\mathcal{L}_G)\to H^{i+1}_c(X_{\text{ét}},\mathcal{L}_G)$ that is induced by the exact triangle in Proposition 4.1(iii). Then, in this setting, the endomorphisms $H^i(1-\phi)$ of $H^i(\mathbb{F}^c_q, s^*_c Rs_{X,!}\mathcal{L}_G)$ are expected to be semisimple at zero under some very general conditions (see, for example, the discussion of Kato in [31, Rem. 3.5.4]) and, in any such case, Proposition 3.2(i) implies that the sequence $\epsilon^{\text{Bock}}_{\mathcal{L},G,\phi}$ is exact.
(ii) An analysis of cup products on the level of complexes as in Rapaport and Zink [39, 1.2] shows that the homomorphisms $\beta^i_{\mathcal{L},G,\phi}$ in (i) have an alternative description (in this regard see also [34, Prop. 6.5], [31, §3.5.2] and, most relevantly, the discussion in [9, §3.2.1]). To be precise, one finds that $\beta^i_{\mathcal{L},G,\phi}$ agrees with the homomorphism $H^i_c(X_{\text{ét}},\mathcal{L}_G)\to H^{i+1}_c(X_{\text{ét}},\mathcal{L}_G)$ that is induced by taking cup-product with the element of $H^1(X_{\text{ét}},\mathbb{Z}_\ell)$ obtained by pulling back the (unique) element of $H^1(\text{Spec}(\mathbb{F}_q)_{\text{ét}},\mathbb{Z}_\ell)=\text{Hom}_{\text{cont}}(\text{Gal}(\mathbb{F}^c_q/\mathbb{F}_q),\mathbb{Z}_\ell)$ that sends ϕ to -1.

4.2. **Zeta functions.** In this section we fix a Galois cover $f: Y\to X$ of group G as in §4.1.1. We also fix a prime ℓ, a finite extension Q of $\mathbb{Q}_\ell$ in $\mathbb{Q}^c_\ell$ with valuation ring O and a flat smooth O-sheaf $\mathcal{L}$ on X as in §4.1.2.

4.2.1. In the sequel we write X^0 for the set of closed points of X. For each x in X^0 we write $\mathrm{d}(x)$ for the degree of x over $\mathbb{F}_q$, we choose an identification of the residue field of x with the subfield $\mathbb{F}_{q^{\mathrm{d}(x)}}$ of $\mathbb{F}^c_q$ and we fix a choice of $\mathbb{F}^c_q$-valued point $\overline{x}$ of X that lies over x.

The stalk $\mathcal{L}_{G,\overline{x}}$ at $\overline{x}$ of the induced sheaf $\mathcal{L}_G$ is then a finitely generated free $O[G]$-module that is equipped with an action of $\phi^{\mathrm{d}(x)}$.

In particular, therefore, we can follow the approach of Deligne in [19, Rem. 2.12] to define (via Galois descent) an element of the power series ring $\zeta(Q[G])[[t]]$ in an indeterminate t over the centre $\zeta(Q[G])$ of $Q[G]$ by setting

$$Z_G(X,\mathcal{L},t) := \prod_{x\in X^0}\text{Nrd}_{Q[G]}(1-\phi^{\mathrm{d}(x)}t^{\mathrm{d}(x)}\mid Q\otimes_O\mathcal{L}_{G,\overline{x}})^{-1}. \tag{23}$$

This is the natural 'equivariant' Zeta function that is attached to the sheaf $\mathcal{L}_G$ of $O[G]$-modules.

Given an element $f=f(t)$ of $\zeta(Q[G])[[t]]$ we write $e_f:\text{Spec}(\zeta(Q[G]))\to\mathbb{Z}$ for its algebraic order at $t=1$. We identify e_f with an element of $\mathbb{Z}^{\pi_0(\text{Spec}(\zeta(Q[G])))}$ in the natural way and then set

$$f^*(t) := (1-t)^{-e_f}\cdot f(t)\in\zeta(Q[G])[[t]].$$

We fix a completion $\mathbb{C}_\ell$ of $\mathbb{Q}^c_\ell$ and use the Wedderburn decomposition of $Q[G]$ to identify $\zeta(Q[G])[[t]]$ as a subring of the product $\prod_\chi\mathbb{C}_\ell[[t]]$, where χ runs over the set $\text{Ir}_\ell(G)$ of irreducible $\mathbb{Q}^c_\ell$-valued characters of G. If the limit

$$f^*(1) := \lim_{t\to 1} f^*(t)$$

is both well-defined in $\prod_\chi\mathbb{C}_\ell$ and also belongs to the subset $\zeta(Q[G])^\times$, then we shall say that the leading term $f^*(1)$ of $f(t)$ at $t=1$ is 'well-defined as an element of $\zeta(Q[G])^\times$'.

We can now state one of the main results of this article. In this result we use the composite homomorphism $\delta_{O[G]}$ that is defined in (15).

Theorem 4.3. *We assume that in each degree i the endomorphism $H^i(1-\phi)$ of the $O[G]$-module $H^i(\mathbb{F}_q^c, s_c^*Rs_{X,!}\mathcal{L}_G)$ is semisimple at zero.*

Then the sequence $\epsilon^{\rm Bock}_{\mathcal{L},G,\phi}$ of $Q[G]$-modules defined in Remark 4.2(i) is exact. In addition, the leading term $Z^*_G(X,\mathcal{L},1)$ is well-defined as an element of $\zeta(Q[G])^\times$ and in $K_0(O[G],Q[G])$ one has

$$\delta_{O[G]}(Z^*_G(X,\mathcal{L},1)) = -\chi_{O[G]}(R\Gamma_c(X_{\rm ét},\mathcal{L}_G),\epsilon^{\rm Bock}_{\mathcal{L},G,\phi}).$$

The proof of this result will be completed in §5 by means of a detailed analysis of the case $\ell = p$. In the remainder of this section we shall reduce the proof of Theorem 4.3 to this special case.

4.2.2. In this section we assume the hypothesis in the first paragraph of Theorem 4.3. At the outset we note that, under this hypothesis, the existence of an exact sequence $\epsilon^{\rm Bock}_{\mathcal{L},G,\phi}$ of $Q[G]$-modules follows directly from the discussion in Remark 4.2(i).

To proceed we fix a finite extension E of Q in $\mathbb{Q}_\ell^c$ over which all characters in ${\rm Ir}_\ell(G)$ can be realised. For each χ in ${\rm Ir}_\ell(G)$ we then also fix a representation $G\to {\rm Aut}_E(V_\chi)$ of character χ.

In each degree i we define finitely generated $Q[G]$-modules

$$V^i := Q\otimes_O H^i(\mathbb{F}_q^c, s_c^*Rs_{X,!}\mathcal{L}_G) \text{ and } V^{i,0} := Q\otimes_O \ker(H^i(1-\phi))\subseteq V^i.$$

Then, as $H^i(1-\phi)$ is assumed to be semisimple at zero, we can fix a direct sum decomposition of $Q[G][H^i(\phi)]$-modules

$$V^i = D^i\oplus V^{i,0}. \tag{24}$$

Thus, in this case, the restriction of $H^i(1-\phi) = 1-H^i(\phi)$ induces an automorphism of the $Q[G]$-module D^i and, by applying the result of Proposition 3.2(ii) to the exact triangle in Proposition 4.1(iii), we can deduce that

$$\begin{aligned}\chi_{O[G]}(R\Gamma_c(X_{\rm ét},\mathcal{L}_G),\epsilon^{\rm Bock}_{\mathcal{L},G,\phi}) &= -\chi_{O[G]}(R\Gamma_c(X_{\rm ét},\mathcal{L}_G)[1],\epsilon^{\rm Bock}_{\mathcal{L},G,\phi})\\ &= -\partial_{O[G],Q[G]}(\textstyle\prod_{i\in\mathbb{Z}}(H^i(1-\phi)^\diamond_Q)^{(-1)^i})\\ &= -\textstyle\sum_{i\in\mathbb{Z}}(-1)^i\delta_{O[G]}({\rm Nrd}_{Q[G]}(1-H^i(\phi)\mid D^i))\\ &= \delta_{O[G]}(\textstyle\prod_{i\in\mathbb{Z}}{\rm Nrd}_{Q[G]}(1-H^i(\phi)\mid D^i)^{(-1)^{i+1}}).\end{aligned}\tag{25}$$

Now in each degree i the decomposition (24) implies that in $\zeta(Q[G])[[t]]$ one has

$$\begin{aligned}{\rm Nrd}_{Q[G]}(1-H^i(\phi)\cdot t : V^i) &= {\rm Nrd}_{Q[G]}(1-H^i(\phi)\cdot t : D^i)\cdot {\rm Nrd}_{Q[G]}(1-t : V^{i,0})\\ &= {\rm Nrd}_{Q[G]}(1-H^i(\phi)\cdot t : D^i)\cdot \textstyle\sum_{\chi\in{\rm Ir}_\ell(G)}(1-t)^{r^i_\chi}e_\chi,\end{aligned}$$

where e_χ is the idempotent $\chi(1)|G|^{-1}\sum_{g\in G}\chi(g^{-1})g$ of $\zeta(\mathbb{Q}_\ell^c[G])$ and we set

$$r^i_\chi := \dim_E({\rm Hom}_{E[G]}(V_{\check\chi}, E\otimes_Q V^{i,0})).$$

where $\check\chi$ denotes the contragredient of χ.

We note also that, since $V^{i,0}$ is a $Q[G]$-module one has $r^i_{\omega\circ\chi} = r^i_\chi$ for all ω in ${\rm Gal}(\mathbb{Q}_\ell^c/Q)$ and so the vector $(r^i_\chi)_{\chi\in{\rm Ir}_\ell(G)}$ corresponds to an element of ${\rm Spec}(\zeta(Q[G]))\to\mathbb{Z}$.

Hence, since $\mathrm{Nrd}_{Q[G]}(1-H^i(\phi):D^i)$ is an element of $\zeta(Q[G])^\times$, this computation implies that $(r^i_\chi)_{\chi\in\mathrm{Ir}_\ell(G)}$ is the algebraic order at $t=1$ of $\mathrm{Nrd}_{Q[G]}(1-H^i(\phi)\cdot t:V^i)$ and that the leading term at $t=1$ of this series is both well-defined as an element of $\zeta(Q[G])^\times$ and equal to $\mathrm{Nrd}_{Q[G]}(1-H^i(\phi):D^i)$.

Taken in conjunction with the equality (25) this fact then implies that the leading term $\xi^*_G(X,\mathcal{L},1)$ at $t=1$ of the series

$$\xi_G(X,\mathcal{L},t):=\prod_{i\in\mathbb{Z}}\mathrm{Nrd}_{Q[G]}(1-\phi\cdot t:V^i)^{(-1)^{i+1}}\in\zeta(Q[G])[[t]] \tag{26}$$

is a well-defined element of $\zeta(Q[G])^\times$ such that in $K_0(O[G],Q[G])$ there is an equality

$$\chi_{O[G]}(R\Gamma_c(X_{\text{ét}},\mathcal{L}_G),\epsilon^{\text{Bock}}_{\mathcal{L},G,\phi})=\delta_{O[G]}(\xi^*_G(X,\mathcal{L},1)).$$

To derive the result of Theorem 4.3 as a consequence of these observations it is therefore enough to prove the following result.

Theorem 4.4. *The leading term $Z^*_G(X,\mathcal{L},1)$ at $t=1$ of $Z_G(X,\mathcal{L},t)$ is well-defined as an element of $\zeta(Q[G])^\times$ and such that in $K_0(O[G],Q[G])$ one has*

$$\delta_{O[G]}(Z^*_G(X,\mathcal{L},1))=\delta_{O[G]}(\xi^*_G(X,\mathcal{L},1)). \tag{27}$$

4.2.3. In this section we fix a prime ℓ with $\ell\neq p$ and use the main result of Grothendieck in [25] to prove that in this case there is an equality of series

$$Z_G(X,\mathcal{L},t)=\xi_G(X,\mathcal{L},t) \tag{28}$$

in $\zeta(Q[G])[[t]]$. This equality combines with the observations about the series $\xi_G(X,\mathcal{L},t)$ that are made above to immediately imply the result of Theorem 4.4 for the prime ℓ, and thereby completes the proof of Theorem 4.3 in this case.

For each character χ in $\mathrm{Ir}_\ell(G)$ we consider the power series

$$Z(X,\mathcal{L}_\chi,t):=\prod_{x\in X^0}\det_E(1-\phi^{\mathrm{d}(x)}t^{\mathrm{d}(x)}\mid \mathrm{Hom}_{E[G]}(V_{\check{\chi}},E\otimes_O\mathcal{L}_{G,\overline{x}}))^{-1}$$

in $E[[t]]$ and in each degree i we define an E-space $V^i(\chi):=\mathrm{Hom}_{E[G]}(V_{\check{\chi}},E\otimes_Q V^i)$.

Then, since the series $Z_G(X,\mathcal{L},t)$ and $\xi_G(X,\mathcal{L},t)$ are each defined via Galois descent, the equality (28) is true if and only if for each χ in $\mathrm{Ir}_\ell(G)$ one has

$$Z(X,\mathcal{L}_\chi,t)=\prod_{i\in\mathbb{Z}}\det_E(1-\phi\cdot t:V^i(\chi))^{(-1)^{i+1}} \tag{29}$$

in $E[[t]]$. Now the isomorphism (22) induces an identification

$$\mathrm{Hom}_{E[G]}(V_{\check{\chi}},E\otimes_O\mathcal{L}_{G,\overline{x}})\cong V_\chi\otimes_O\mathcal{L}_{\overline{x}},$$

and so $Z(X,\mathcal{L}_\chi,t)$ is the Zeta function that is naturally associated to the constructible sheaf of E-vector spaces $V_\chi\otimes_O\mathcal{L}_{\overline{x}}$ on X.

Since each vector space $V^i(\chi)$ identifies with $H^i_c((\mathbb{F}^c_q\times_{\mathbb{F}_q}X)_{\text{ét}},V_\chi\otimes_O\mathcal{L}_{\overline{x}})$, the required equality (29) therefore follows directly from the exposition of Grothendieck's results that is given by Milne in [33, Chap. VI, Th. 13.3].

5. p-ADIC SHEAVES

In this section we shall combine Corollary 2.3 with a result of Emerton and Kisin in order to prove Theorem 4.4 in the case $\ell = p$, thereby also completing the proof of Theorem 4.3.

We therefore continue to use the same notation as in §4.2.

5.1. **Preliminary constructions.** For each $O[G]$-module M we write M_t for the associated $O[G][[t]]$-module $O[[t]]\otimes_O M$.

Then for an endomorphism η of M and any series in $O[\eta][[t]]$ of the form

$$\varpi(\eta,t) := 1 + \sum_{j\geq 1}\sum_{i\geq 0} m_{ij}\eta^i t^j$$

one obtains a well-defined automorphism of M_t by setting

$$\varpi(\eta,t)(\lambda\otimes m) := \lambda\otimes m + \sum_{j\geq 1}\sum_{i\geq 0} m_{ij}t^j\lambda\otimes\eta^i(m)$$

for each $\lambda\in O[[t]]$ and $m\in M$.

In particular, if M is a finitely generated projective $O[G]$-module, then M_t is a finitely generated projective $O[G][[t]]$-module and the pair $(\varpi(\eta,t), M_t)$ gives rise to a well-defined element $\langle\varpi(\eta,t)\mid M_t\rangle$ of $K_1(O[G][[t]])$.

In this way we may associate an element of $K_1(O[G][[t]])$ to the given sheaf of O-modules $\mathcal{L}$ on X by setting

$$\mathcal{Z}_G(X,\mathcal{L},t) := \langle\textstyle\prod_{x\in X^0}(1-\phi^{\mathrm{d}(x)}t^{\mathrm{d}(x)})^{-1}\mid(\mathcal{L}_{G,\overline{x}})_t\rangle.$$

Next we fix (following Proposition 4.1(ii)) a complex $P^\bullet$ in $C^{\mathrm{perf}}(O[G])$ and an endomorphism $\hat\phi$ of $P^\bullet$ for which there exists a commutative diagram in $D^{\mathrm{perf}}(O[G])$

$$\begin{array}{ccc} P^\bullet & \xrightarrow{\ \iota\ } & R\Gamma(\mathbb{F}_q^c, s_c^*Rs_{X,!}\mathcal{L}_G) \\ {\scriptstyle 1-\hat\phi}\downarrow & & \downarrow{\scriptstyle 1-\phi} \\ P^\bullet & \xrightarrow{\ \iota\ } & R\Gamma(\mathbb{F}_q^c, s_c^*Rs_{X,!}\mathcal{L}_G)\end{array} \tag{30}$$

in which ι is an isomorphism. Then, just as above, each such $\hat\phi$ gives rise to a well-defined element

$$\mathcal{Z}_G(1-\hat\phi,t) := \prod_{i\in\mathbb{Z}}\langle 1-\hat\phi^i t\mid P^i_t\rangle^{(-1)^{i+1}}$$

of $K_1(O[G][[t]])$.

For each subgroup U of G, we consider the composite homomorphism

$$\theta^t_U : K_1(O[G][[t]])\to K_1(\mathcal{O}[U][[t]])\to K_1(\mathcal{O}[U^{\mathrm{ab}}][[t]])\xrightarrow{\mathrm{det}^t_{O[U^{\mathrm{ab}}]}}\mathcal{O}[U^{\mathrm{ab}}][[t]]^\times,$$

where the first map is given by restriction of scalars, the second is the natural projection and $\mathrm{det}^t_{O[U^{\mathrm{ab}}]}$ denotes the isomorphism induced by taking determinants over $\mathcal{O}[U^{\mathrm{ab}}][[t]]$.

In the following result we describe the images of $\mathcal{Z}_G(X,\mathcal{L},t)$ and $\mathcal{Z}_G(1-\hat\phi,t)$ under each of the maps θ^t_U. In this result we use the notation for Galois covers as in diagram (21).

Proposition 5.1. *For each subgroup U of G, the following claims are valid.*

(i) $\theta^t_U(\mathcal{Z}_G(X,\mathcal{L},t)) = Z_{U^{\rm ab}}(Y^U, f_U^*\mathcal{L},t)$.

(ii) *Write $P^\bullet_{U^{\rm ab}}$ for the complex $O[U^{\rm ab}]\otimes_{O[U]} P^\bullet$ in $C^{\rm perf}(O[U^{\rm ab}])$. Then $\hat\phi$ induces an endomorphism $\hat\phi_{U^{\rm ab}}$ of $P^\bullet_{U^{\rm ab}}$ that lies in a commutative diagram in $D^{\rm perf}(O[U^{\rm ab}])$*

$$\begin{array}{ccc} P^\bullet_{U^{\rm ab}} & \xrightarrow{\iota'} & R\Gamma(\mathbb{F}_q^c, s_c^*Rs_{Y^U,!}(f_U^*\mathcal{L})_{U^{\rm ab}}) \\ {\scriptstyle 1-\hat\phi_{U^{\rm ab}}}\downarrow & & \downarrow{\scriptstyle 1-\phi} \\ P^\bullet_{U^{\rm ab}} & \xrightarrow{\iota'} & R\Gamma(\mathbb{F}_q^c, s_c^*Rs_{Y^U,!}(f_U^*\mathcal{L})_{U^{\rm ab}}) \end{array}$$

in which ι' is an isomorphism, and one has

$$\theta^t_U(\mathcal{Z}_G(1-\hat\phi,t)) = {\rm det}^t_{O[U^{\rm ab}]}(\mathcal{Z}_{U^{\rm ab}}(1-\hat\phi_{U^{\rm ab}},t)).$$

Proof. For an endomorphism α of a (complex of) $O[G]$-modules M we write $\alpha_{U^{\rm ab}}$ for the induced endomorphism ${\rm id}\otimes\alpha$ of $M_{U^{\rm ab}} := O[U^{\rm ab}]\otimes_{O[U]} M$.

Then the equality in claim (i) is valid because

$$\begin{aligned} \theta^t_{U^{\rm ab}}(\mathcal{Z}_G(X,\mathcal{L},t)) &= {\rm det}^t_{O[U^{\rm ab}]}(\langle \textstyle\prod_{x\in X^0}(1-\phi_{U^{\rm ab}}^{{\rm d}(x)}t^{{\rm d}(x)})^{-1} \mid (\mathcal{L}_{G,\overline{x}})_{U^{\rm ab},t}\rangle) \\ &= \prod_{x\in X^0} {\rm det}^t_{O[U^{\rm ab}]}(\langle 1-\phi_{U^{\rm ab}}^{{\rm d}(x)}t^{{\rm d}(x)} \mid (\mathcal{L}_{G,\overline{x}})_{U^{\rm ab},t}\rangle)^{-1} \\ &= \prod_{x\in X^0} {\rm det}_{O[U^{\rm ab}]}(1-\phi_{U^{\rm ab}}^{{\rm d}(x)}t^{{\rm d}(x)} \mid (\mathcal{L}_{G,\overline{x}})_{U^{\rm ab}})^{-1} \\ &= \prod_{y\in (Y^U)^0} {\rm det}_{O[U^{\rm ab}]}(1-\phi_{U^{\rm ab}}^{{\rm d}(y)}t^{{\rm d}(y)} \mid (f_U^*\mathcal{L})_{U^{\rm ab},\overline{y}})^{-1} \\ &= Z_{U^{\rm ab}}(Y^U, f_U^*\mathcal{L},t). \end{aligned}$$

All equalities in this display are clear except for the fourth, which follows from a slight variant of a standard argument. To explain this point we set $Y' := Y^U$ and write $\mathcal{L}'$ for the sheaf of O-modules $f_U^*\mathcal{L}$ on Y'. For each x in X^0 we also write $Y'(x)$ for the subset of $(Y')^0$ comprising those points that f_U maps to x. Then, to verify the fourth equality, it is enough to show that for each such point x one has

$$(31)\qquad {\rm det}_{O[U^{\rm ab}]}(1-\phi^{{\rm d}(x)}t^{{\rm d}(x)} \mid (\mathcal{L}_{G,\overline{x}})_{U^{\rm ab}}) = \prod_{y\in Y'(x)} {\rm det}_{O[U^{\rm ab}]}(1-\phi^{{\rm d}(y)}t^{{\rm d}(y)} \mid \mathcal{L}'_{U^{\rm ab},\overline{y}}).$$

Now there is a natural isomorphism $f_U^*(\mathcal{L}_G)\cong O[G]\otimes_{O[U]}(\mathcal{L}')_U$ of sheaves on Y' and for any $\overline{y}_0$ in $F_{\overline{x}}(Y')$ the map $g\mapsto g(\overline{y}_0)$ induces a bijection between the set of cosets of U in G and the set $F_{\overline{x}}(Y')$. There is therefore a decomposition

$$\mathcal{L}_{G,\overline{x}} \cong (f_U^*\mathcal{L}_G)_{\overline{y}_0} \cong \bigoplus_{\overline{y}\in F_{\overline{x}}(Y')} (\mathcal{L}')_{U,\overline{y}}.$$

Further, if y is the point of Y' that lies below $\overline{y}$ and we set $e_{y/x} := {\rm d}(y)/{\rm d}(x)$, then the action of $\phi_x := \phi^{{\rm d}(x)}$ induces isomorphisms of $O[U]$-modules $\mathcal{L}'_{U,\phi_x^m(\overline{y})}\cong \mathcal{L}'_{U,\overline{y}}$ for each integer m

with $1 \le m \le e_{y/x}$. In particular, one has $\phi_x^{e_{y/x}}(\overline{y}) = \overline{y}$ and the isomorphism of $\mathcal{L}'_{U,\phi_x^{e_{y/x}}(\overline{y})}$ with $\mathcal{L}'_{U,\overline{y}}$ is induced by the action of $\phi_x^{e_{y/x}} = \phi^{\mathrm{d}(y)}$.

One can therefore decompose $(\mathcal{L}_{G,\overline{x}})_{U^{\mathrm{ab}}}$ as a direct sum

$$(\mathcal{L}_{G,\overline{x}})_{U^{\mathrm{ab}}} = \bigoplus_{y\in Y'(x)} \bigoplus_{F_{\overline{x}}(y)} (\mathcal{L}'_{U,\overline{y}})_{U^{\mathrm{ab}}} \cong \bigoplus_{y\in Y'(x)} \bigoplus_{F_{\overline{x}}(y)} \mathcal{L}'_{U^{\mathrm{ab}},\overline{y}}, \tag{32}$$

where $F_{\overline{x}}(y)$ is the set of geometric points of Y' over y and $\overline{y}$ is a choice of element of $F_{\overline{x}}(y)$. Further, the endomorphism of $\bigoplus_{F_{\overline{x}}(y)}\mathcal{L}'_{U^{\mathrm{ab}},\overline{y}}$ that is induced by $1-\phi_{U^{\mathrm{ab}}}^{\mathrm{d}(x)}t^{\mathrm{d}(x)}$ is represented by the $e_{y/x}\times e_{y/x}$ matrix

$$\begin{pmatrix} 1 & 0 & \dots & 0 & -\phi_{U^{\mathrm{ab}}}^{\mathrm{d}(y)}t^{\mathrm{d}(x)} \\ -t^{\mathrm{d}(x)} & 1 & 0 & \dots & 0 \\ 0 & -t^{\mathrm{d}(x)} & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & -t^{\mathrm{d}(x)} & 1 \end{pmatrix}.$$

By using elementary row operations this matrix can be reduced to an upper triangular matrix in which the last diagonal entry is equal to

$$1-\phi_{U^{\mathrm{ab}}}^{\mathrm{d}(y)}(t^{\mathrm{d}(x)})^{e_{y/x}} = 1-\phi_{U^{\mathrm{ab}}}^{\mathrm{d}(y)}t^{\mathrm{d}(y)}$$

and all other diagonal entries are equal to 1 and so one has

$$\det_{O[U^{\mathrm{ab}}]}(1-\phi^{\mathrm{d}(x)}t^{\mathrm{d}(x)} \mid \bigoplus_{F_{\overline{x}}(y)}\mathcal{L}'_{U^{\mathrm{ab}},\overline{y}}) = \det_{O[U^{\mathrm{ab}}]}(1-\phi^{\mathrm{d}(y)}t^{\mathrm{d}(y)} \mid (\mathcal{L}'_{U^{\mathrm{ab}}})_{\overline{y}}).$$

The claimed equality (31) thus follows from the decomposition (32), as required to prove claim (i).

The existence of a commutative diagram as in claim (ii) is a direct consequence of the (well-known) fact that the complex $O[U^{\mathrm{ab}}]\otimes^{\mathbb{L}}_{O[U]} R\Gamma(\mathbb{F}_q^c, s_c^*Rs_{X,!}\mathcal{L}_G)$ is isomorphic in $D^{\mathrm{perf}}(O[U^{\mathrm{ab}}])$ to $R\Gamma(\mathbb{F}_q^c, s_c^*Rs_{Y^U,!}(f_U^*\mathcal{L})_{U^{\mathrm{ab}}})$. In addition, the second assertion of claim (ii) is true because

$$\begin{aligned} \theta_U^t(\mathcal{Z}_G(1-\hat{\phi},t)) &= \prod_{i\in\mathbb{Z}} \theta_U^t(\langle 1-\hat{\phi}^i t \mid P_t^i\rangle)^{(-1)^{i+1}} \\ &= \prod_{i\in\mathbb{Z}} \det{}^t_{O[U^{\mathrm{ab}}]}(\langle 1-\hat{\phi}^i_{U^{\mathrm{ab}}} t \mid P^i_{U^{\mathrm{ab}},t}\rangle)^{(-1)^{i+1}} \\ &= \det{}^t_{O[U^{\mathrm{ab}}]}\big(\mathcal{Z}_{U^{\mathrm{ab}}}(1-\hat{\phi}_{U^{\mathrm{ab}}},t)\big). \end{aligned}$$

Here the second equality is true because for each integer i one has

$$\theta_{U,1}^t(\langle 1-\hat{\phi}^i t \mid P_t^i\rangle) = \langle 1-(\mathrm{id}\otimes\hat{\phi}^i)t \mid (O[U^{\mathrm{ab}}]\otimes_{O[U]} P^i)_t\rangle,$$

where $\theta_{U,1}^t$ denotes the composition of the first two maps that occur in the definition of θ_U^t. In addition, the first and third equalities follow directly from the respective definitions of $\mathcal{Z}_G(1-\hat{\phi},t)$ and $\mathcal{Z}_{U^{\mathrm{ab}}}(1-\hat{\phi}_{U^{\mathrm{ab}}},t)$. □

5.2. **The proof of Theorem 4.3.**

5.2.1. We first establish an important consequence of Corollary 2.3 and the main result of Emerton and Kisin in [20].

Proposition 5.2. *For any endomorphism $\hat{\phi}$ of complexes of $O[G]$-modules as in (30) the image of $\mathcal{Z}_G(X, \mathcal{L}_G, t) \cdot \mathcal{Z}_G(1-\hat{\phi}, t)^{-1}$ in $K_1'(O[[t]][G])$ belongs to the subgroup $K_1'(O\langle t\rangle[G])$.*

Proof. After taking account of Corollary 2.3, it is enough to show that for each subgroup U of G the element $\theta_U^t(\mathcal{Z}_G(X, \mathcal{L}_G, t)) \cdot \theta_U^t(\mathcal{Z}_G(1-\hat{\phi}, t))^{-1}$ of $O[[t]][U^{\rm ab}]^\times$ belongs to the subgroup $O\langle t\rangle[U^{\rm ab}]^\times$.

In view of Proposition 5.1, it is thus enough to show that for each such U one has

$$Z_{U^{\rm ab}}(Y^U, f_U^*\mathcal{L}, t) \cdot {\rm det}^t_{O[U^{\rm ab}]}(\mathcal{Z}_{U^{\rm ab}}(1-\hat{\phi}_{U^{\rm ab}}, t))^{-1} \in O\langle t\rangle[U^{\rm ab}]^\times. \tag{33}$$

We shall now deduce this from the main result of [20]. To do this we set $\mathcal{F} := (f_U^*\mathcal{L})_{U^{\rm ab}}$ and write $\mathcal{F}^\bullet$ for the complex that is equal to $\mathcal{F}$ in degree zero and is equal to zero in all other degrees.

Then the function $L(Y^U, \mathcal{F}^\bullet, t)$ that is defined in [20, (1.7)] is, by its very definition, equal to $Z_{U^{\rm ab}}(Y^U, f_U^*\mathcal{L}, t)$.

Further, if $\overline{z}$ denotes the geometric point ${\rm Spec}(\mathbb{F}_q^c)$ of ${\rm Spec}(\mathbb{F}_q)$, then [33, Chap. II, Th. 3.2(a)] (with $\pi = s_c$) implies that the complex of stalks $(Rs_{Y^U,!}\mathcal{F})^\bullet_{\overline{z}}$ identifies with $R\Gamma(\mathbb{F}_q^c, s_c^*Rs_{Y^U,!}\mathcal{F})$. By using the result of [18, p. 115, Cor. 1.13], we can therefore deduce from the commutative diagram in Proposition 5.1(ii) that the series $L(\mathbb{F}_q, Rs_{Y^U,!}\mathcal{F}^\bullet, t)$ defined in [20] coincides with ${\rm det}^t_{O[U^{\rm ab}]}(\mathcal{Z}_{U^{\rm ab}}(1-\hat{\phi}_{U^{\rm ab}}, t))$.

Given these observations, and the fact that $O\langle t\rangle[U^{\rm ab}]^\times$ contains $1 + tO\langle t\rangle[U^{\rm ab}]$, the required containment (33) follows directly from the containment

$$L(Y^U, \mathcal{F}^\bullet, t) \cdot L(\mathbb{F}_q, Rs_{Y^U,!}\mathcal{F}^\bullet, t)^{-1} \in 1 + tO\langle t\rangle[U^{\rm ab}]$$

that is proved by Emerton and Kisin in [20, Cor. 1.8]. □

5.2.2. In the next result we fix (as we may) a finite extension E of Q in $\mathbb{Q}_p^c$, with valuation ring O_E, such that for every χ in ${\rm Ir}_p(G)$ there exists an $O_E[G]$-module that is free over O_E and spans an $E[G]$-module V_χ (as in §4.2.2 with $\ell = p$) of character χ.

Lemma 5.3. *There exists a canonical 'reduced norm' homomorphism of abelian groups*

$${\rm Nrd}_{O[[t]][G]} : K_1'(O[[t]][G]) \to \prod_{{\rm Ir}_p(G)} O_E[[t]]^\times$$

that has both of the following properties.

(i) *If α is an automorphism of a finitely generated projective $O[[t]][G]$-module P, then*

$${\rm Nrd}_{O[[t]][G]}(\langle \alpha \mid P\rangle) = \left({\rm det}_{E[[t]]}(\alpha \mid {\rm Hom}_{E[G]}(V_{\check{\chi}}, E \otimes_O P))\right)_{\chi\in{\rm Ir}_p(G)}.$$

(ii) *For any element η of $K_1'(O\langle t\rangle[G])$ the tuple ${\rm Nrd}_{O[[t]][G]}(\eta)$ converges at $t = 1$ to give an element of the subgroup ${\rm Nrd}_{Q[G]}(K_1(O[G]))$ of $\zeta(Q[G])^\times \subset \prod_{{\rm Ir}_p(G)} E^\times$.*

Proof. The choice of E ensures that each character χ in $\mathrm{Ir}_p(G)$ corresponds to ring homomorphism $\rho_\chi : O[G] \to \mathrm{M}_{\chi(1)}(O_E)$. For each natural number n, each matrix $M = (M_{ij})$ in $\mathrm{GL}_n(O[[t]][G])$ and each χ in $\mathrm{Ir}_p(G)$, we write $\rho_\chi(M)$ for the matrix in $\mathrm{M}_{n\chi(1)}(O_E[[t]])$ that is obtained by applying ρ_χ to the coefficients of powers of t in every component M_{ij} of M.

By sending each M to $(\det(\rho_\chi(M)))_\chi$ one then obtains a homomorphism

$$\nu_n : \mathrm{GL}_n(O[[t]][G]) \to \prod_{\mathrm{Ir}_p(G)} O_E[[t]]^\times$$

such that $\nu_{n+1} \circ \iota_n = \nu_n$, with ι_n the standard inclusion $\mathrm{GL}_n(O[[t]][G]) \to \mathrm{GL}_{n+1}(O[[t]][G])$. The maps $(\nu_n)_n$ therefore induce a homomorphism

$$\mathrm{GL}(O[[t]][G]) \to \prod_{\mathrm{Ir}_p(G)} O_E[[t]]^\times$$

that clearly factorizes through the abelianization $K_1(O[[t]][G])$ of $\mathrm{GL}(O[[t]][G])$. It is also clear that the latter homomorphism factors through the natural map $K_1(O[[t]][G]) \to K_1(O[[t]][1/p][G])$ and so induces a homomorphism

$$\mathrm{Nrd}_{O[[t]][G]} : K_1'(O[[t]][G]) \to \prod\nolimits_{\mathrm{Ir}_p(G)} O_E[[t]]^\times$$

of the required sort.

With this definition of $\mathrm{Nrd}_{O[[t]][G]}$, the property in claim (i) is then a consequence of the fact that for M in $\mathrm{GL}_n(O[[t]][G])$, and χ in $\mathrm{Ir}_p(G)$, the χ-component of the image under $\mathrm{Nrd}_{O[[t]][G]}$ of (the image in $K_1'(O[[t]][G])$ of) M is equal to

$$\det(\rho_\chi(M)) = \det_{E[[t]]}(\varrho_M \mid \mathrm{Hom}_{E[G]}(V_{\check\chi}, E \otimes_O O[[t]][G]^n))$$

where ϱ_M denotes the automorphism of $E \otimes_O O[[t]][G]^n$ that corresponds to M.

Next we note that if M is a matrix in $\mathrm{GL}_n(O\langle t\rangle[G])$, then the value $M(1)$ of M at $t = 1$ belongs to $\mathrm{GL}_n(O[G])$ and for each χ in $\mathrm{Ir}_p(G)$ the matrix $\rho_\chi(M)$ belongs to $\mathrm{GL}_{n\chi(1)}(O_E\langle t\rangle)$. For each such M the tuple $(\det(\rho_\chi(M)))_\chi$ therefore belongs to $\prod_\chi O_E\langle t\rangle$ and converges at $t = 1$ to the element $(\det(\rho_\chi(M(1)))_\chi = \mathrm{Nrd}_{Q[G]}(M(1))$ of $\mathrm{Nrd}_{Q[G]}(K_1(O[G]))$. Claim (ii) follows directly from this fact. □

Turning now to the proof of Theorem 4.3, we note Lemma 5.3(i) implies that for each x in X^0 one has

$$\begin{aligned}
&\mathrm{Nrd}_{O[[t]][G]}\big(\langle 1 - \phi^{\mathrm{d}(x)}t^{\mathrm{d}(x)} \mid (\mathcal{L}_{G,\overline{x}})_t\rangle\big)\\
&= \big(\det_{E[[t]]}(\langle 1 - \phi^{\mathrm{d}(x)}t^{\mathrm{d}(x)} \mid \mathrm{Hom}_{E[G]}(V_{\check\chi}, E \otimes_O (\mathcal{L}_{G,\overline{x}})_t)\big)_{\chi\in\mathrm{Ir}_p(G)}\\
&= \big(\det_{E}(\langle 1 - \phi^{\mathrm{d}(x)}t^{\mathrm{d}(x)} \mid \mathrm{Hom}_{E[G]}(V_{\check\chi}, E \otimes_O (\mathcal{L}_{G,\overline{x}}))\big)_{\chi\in\mathrm{Ir}_p(G)}\\
&= \mathrm{Nrd}_{Q[G]}(1 - \phi^{\mathrm{d}(x)}t^{\mathrm{d}(x)} \mid Q \otimes_O \mathcal{L}_{G,\overline{x}})
\end{aligned}$$

and hence that $\mathrm{Nrd}_{O[[t]][G]}(\mathcal{Z}_G(X, \mathcal{L}, t)) = Z_G(X, \mathcal{L}, t)$.

In a similar way, one has

$$
\begin{aligned}
\mathrm{Nrd}_{O[[t]][G]}(\mathcal{Z}_G(1-\hat{\phi},t)) &= \prod_{i\in\mathbb{Z}} \mathrm{Nrd}_{O[[t]][G]}\big(\langle 1-\hat{\phi}^i t \mid P_t^i\rangle\big)^{(-1)^{i+1}} \\
&= \prod_{i\in\mathbb{Z}} \mathrm{Nrd}_{Q[G]}\big(1-\hat{\phi}^i t \mid Q\otimes_O P^i\big)^{(-1)^{i+1}} \\
&= \prod_{i\in\mathbb{Z}} \mathrm{Nrd}_{Q[G]}\big(1-H^i(\hat{\phi})t \mid Q\otimes_O H^i(P^\bullet)\big)^{(-1)^{i+1}} \\
&= \prod_{i\in\mathbb{Z}} \mathrm{Nrd}_{Q[G]}\big(1-H^i(\phi)t \mid Q\otimes_O H^i(\mathbb{F}_q^c, s_c^* Rs_{X,!}\mathcal{L}_G)\big)^{(-1)^{i+1}} \\
&= \xi_G(X,\mathcal{L},t).
\end{aligned}
$$

Here the third equality follows from the fact $Q[G]$ is semisimple and the fourth from the commutativity of diagram (30), whilst the last is a direct consequence of the definition (26) of $\xi_G(X,\mathcal{L},t)$.

In view of Proposition 5.2, we can therefore deduce from Lemma 5.3(ii) that the leading term at $t=1$ of

$$
\begin{aligned}
Z_G(X,\mathcal{L},t)\cdot \xi_G(X,\mathcal{L},t)^{-1} &= \mathrm{Nrd}_{O[[t]][G]}(\mathcal{Z}_G(X,\mathcal{L},t))\cdot \mathrm{Nrd}_{O[[t]][G]}(\mathcal{Z}_G(1-\hat{\phi},t))^{-1} \\
&= \mathrm{Nrd}_{O[[t]][G]}(\mathcal{Z}_G(X,\mathcal{L},t)\cdot \mathcal{Z}_G(1-\hat{\phi},t)^{-1})
\end{aligned}
$$

is well-defined as an element of $\zeta(Q[G])^\times$ and belongs to the subgroup $\mathrm{Nrd}_{Q[G]}(K_1(O[G]))$.

We also recall that, by the argument in §4.2.2, the leading term $\xi_G^*(X,\mathcal{L},1)$ at $t=1$ of $\xi_G(X,\mathcal{L},t)$ is well-defined as an element of $\zeta(Q[G])^\times$.

These two observations combine to imply that the leading term $Z_G^*(X,\mathcal{L},1)$ at $t=1$ of $Z_G(X,\mathcal{L},t)$ is both well-defined as an element of $\zeta(Q[G])^\times$ and differs from $\xi_G^*(X,\mathcal{L},1)$ by an element of $\mathrm{Nrd}_{Q[G]}(K_1(O[G]))$.

In particular, since the relevant case of the exact sequence of relative algebraic K-theory (13) implies $\mathrm{Nrd}_{Q[G]}(K_1(O[G]))$ lies in the kernel of the homomorphism $\delta_{O[G]}$, the latter fact implies that $\delta_{O[G]}(Z_G^*(X,\mathcal{L},1)) = \delta_{O[G]}(\xi_G^*(X,\mathcal{L},1))$.

This completes the proof of Theorem 4.4 in the case $\ell = p$, and hence also completes the proof of Theorem 4.3.

Remark 5.4. In the special case that $O=\mathbb{Z}_p$ one can also deduce the validity of Theorem 4.4 (in the case $\ell=p$) by using the main result (Theorem 5.1) of Witte in [50] in place of Proposition 5.2 and Lemma 5.3. See also Remark 2.4 in this regard.

Part II: Weil-étale cohomology and Artin L-series

6. The proofs of Theorem 1.1 and Corollary 1.4

At the outset, we recall for the reader's convenience that Lichtenbaum has observed in [32, §2] that the apparatus of the mapping cylinder category works for the Weil-étale topology in exactly the same way as for the étale topology (as described by Milne in [33, pp. 73-77]). In particular, if Z is a closed subscheme of X with open complement U, and i and j are the immersions from Z and U into X, then the six standard functors $i^*, i_*, i^!, j_!, j^*$ and j_*

exist for the Weil-étale topology (with i_* right adjoint to i^* and left adjoint to $i^!$ and j^* right adjoint to $j_!$ and left adjoint to j_*) and are such that $i^* \circ i_* = \mathrm{id}$ and $i^* \circ j_! = 0$.

6.1. **The proof of Theorem 1.1.** Throughout this section we shall use the notation and hypotheses of Theorem 1.1.

6.1.1. We start by making several technical observations about the Cartesian square (2).

Lemma 6.1. *The following claims are valid.*

(i) *There exists a natural isomorphism in* $D(\mathbb{Z}[G])$ *of the form*

$$R\Gamma(Y'_{W\text{ét}}, j_{Y,!}\mathbb{Z}) \cong R\Gamma(X'_{W\text{ét}}, f'_* f'^* j_!\mathbb{Z}).$$

(ii) *For every prime* ℓ *there exists a natural isomorphism in* $D^-(\mathbb{Z}_\ell[G])$ *of the form*

$$\mathbb{Z}_\ell[G] \otimes^{\mathbb{L}}_{\mathbb{Z}[G]} R\Gamma(Y'_{W\text{ét}}, j_{Y,!}\mathbb{Z}) \cong R\Gamma_c(X_{\text{ét}}, f_* f^* \mathbb{Z}_\ell).$$

Proof. One has $\mathbb{Z} = f^*\mathbb{Z}$ on $Y_{W\text{ét}}$ and $j_{Y,!}f^* = f'^* j_!$ on $X_{W\text{ét}}$. Claim (i) is therefore true since f'_* is exact (as f' is finite) and so there is a natural isomorphism $R\Gamma(Y'_{W\text{ét}}, f'^* j_!\mathbb{Z}) \cong R\Gamma(X'_{W\text{ét}}, f'_* f'^* j_!\mathbb{Z})$ in $D(\mathbb{Z}[G])$.

To prove claim (ii) we note that the functor $\mathbb{Z}_\ell[G] \otimes_{\mathbb{Z}[G]} -$ identifies with the exact functor $\mathbb{Z}_\ell \otimes_{\mathbb{Z}} -$ and that the very definition of compactly supported étale cohomology ensures that $R\Gamma_c(X_{\text{ét}}, f_* f^* \mathbb{Z}_\ell)$ identifies with $R\Gamma(X'_{\text{ét}}, j_! f_* f^* \mathbb{Z}_\ell)$.

In view of claim (i) it is therefore enough to prove that there exists a natural isomorphism in $D(\mathbb{Z}_\ell[G])$ of the form $\mathbb{Z}_\ell \otimes_{\mathbb{Z}} R\Gamma(X'_{W\text{ét}}, f'_* f'^* j_!\mathbb{Z}) \cong R\Gamma(X_{\text{ét}}, j_! f_* f^* \mathbb{Z}_\ell)$.

To construct such an isomorphism we define sheaves of G-modules on $X'_{W\text{ét}}$ by setting $\mathcal{L} := f'_* f'^* j_!\mathbb{Z}$ and $\mathcal{L}/\ell^n := f'_* f'^* j_!(\mathbb{Z}/\ell^n)$ for each natural number n. We also fix an object $P^\bullet$ of $C^-(\mathbb{Z}[G])$ that is isomorphic in $D^-(\mathbb{Z}[G])$ to $R\Gamma(X'_{\text{Wét}}, \mathcal{L})$ and an object $Q^\bullet$ of $C^-(\mathbb{Z}_\ell[G])$ that is isomorphic in $D(\mathbb{Z}_\ell[G])$ to $R\Gamma(X_{\text{ét}}, j_! f_* f^* \mathbb{Z}_\ell)$.

For each n we set $\Lambda_n := \mathbb{Z}/\ell^n[G]$. Then the short exact sequence of complexes

$$0 \to P^\bullet \xrightarrow{\times \ell^n} P^\bullet \to P^\bullet/\ell^n \to 0$$

combines with the natural exact triangle

$$R\Gamma(X'_{\text{Wét}}, \mathcal{L}) \xrightarrow{\times \ell^n} R\Gamma(X'_{\text{Wét}}, \mathcal{L}) \to R\Gamma(X'_{\text{Wét}}, \mathcal{L}/\ell^n) \to R\Gamma(X'_{\text{Wét}}, \mathcal{L})[1]$$

to give an isomorphism $P^\bullet/\ell^n \cong R\Gamma(X'_{\text{Wét}}, \mathcal{L}/\ell^n)$ in $D^{\text{perf}}(\Lambda_n)$. In a similar way there is an isomorphism $Q^\bullet/\ell^n \cong R\Gamma(X_{\text{ét}}, j_! f_* f^*(\mathbb{Z}/\ell^n))$ in $D(\Lambda_n)$.

Next we note that there are natural isomorphisms in $D(\Lambda_n)$ of the form

$$R\Gamma(X'_{\text{Wét}}, \mathcal{L}/\ell^n) \cong R\Gamma(X'_{\text{ét}}, \mathcal{L}/\ell^n) \cong R\Gamma(X'_{\text{ét}}, j_! f_* f^*(\mathbb{Z}/\ell^n)).$$

Here the first isomorphism is by [32, Prop. 2.4(g)], whilst the second is induced by the fact that $f'_* j_{Y,!}\mathcal{T} = j_! f_* \mathcal{T}$ for any torsion sheaf $\mathcal{T}$ on $Y_{\text{Wét}}$ and hence that $\mathcal{L}/\ell^n = f'_* f'^* j_!(\mathbb{Z}/\ell^n) = f'_* j_{Y,!} f^*(\mathbb{Z}/\ell^n)$ is equal to $j_! f_* f^*(\mathbb{Z}/\ell^n)$.

These observations combine to imply the existence of a natural isomorphism $\alpha_n : Q^\bullet/\ell^n \cong P^\bullet/\ell^n$ in $D^-(\Lambda_n)$ and, as n varies, these isomorphisms are such that the diagram

$$\begin{array}{ccccccc} \cdots & \longrightarrow & Q^\bullet/\ell^n & \xrightarrow{\pi_n^Q} & Q^\bullet/\ell^{n-1} & \longrightarrow & \cdots \\ & & \alpha_n\big\downarrow & & \alpha_{n-1}\big\downarrow & & \\ \cdots & \longrightarrow & P^\bullet/\ell^n & \xrightarrow{\pi_n^P} & P^\bullet/\ell^{n-1} & \longrightarrow & \cdots \end{array}$$

commutes in $D^-(\Lambda_n)$, where π_n^Q and π_n^P are the natural quotient maps.

Now, since $Q^\bullet/\ell^n$ consists of projective Λ_n-modules and $P^\bullet/\ell^n$ of Λ_n-modules, one can realize each α_n as an actual map of complexes. Moreover, $\alpha_{n-1}\circ\pi_n^Q$ will then be homotopic to $\pi_n^P\circ\alpha_n$ so that there exists a map $h : Q^\bullet/\ell^n \to P^\bullet/\ell^{n-1}[-1]$ for which one has

$$\alpha_{n-1}\circ\pi_n^Q - \pi_n^P\circ\alpha_n = d\circ h + h\circ d.$$

But, in each degree i, the projection map $P^i/\ell^n \to P^i/\ell^{n-1}$ is surjective and Q^i/ℓ^n is a projective Λ_n-module and so h can be lifted to a map $h' : Q^\bullet/\ell^n \to P^\bullet/\ell^n[-1]$. If one then replaces α_n by $\alpha_n + d\circ h' + h'\circ d$, the above diagram will actually be a commutative diagram of maps of complexes.

By means of an inductive construction, we can therefore assume that, taken together, the maps α_n constitute a map of inverse systems of complexes. By passing to the inverse limit of such a compatible system one then obtains an isomorphism in $D^-(\mathbb{Z}_\ell[G])$ of the form

$$\begin{aligned} R\Gamma(X_{\text{ét}}, j_!f_*f^*\mathbb{Z}_\ell) \cong Q^\bullet \cong \varprojlim_n Q^\bullet/\ell^n \cong \varprojlim_n P^\bullet/\ell^n \\ \cong \mathbb{Z}_\ell\otimes_{\mathbb{Z}} P^\bullet \cong \mathbb{Z}_\ell\otimes_{\mathbb{Z}} R\Gamma(X'_{W\text{ét}}, f'_*f'^*j_!\mathbb{Z}), \end{aligned}$$

as required. □

In the next result we use the sequence of $\mathbb{Q}_\ell[G]$-modules $\epsilon^{\text{Bock}}_{\mathbb{Z}_\ell,G,\phi}$ that is defined in Remark 4.2(i) (with $O = \mathbb{Z}_\ell$).

Lemma 6.2. *Under the hypotheses of Theorem 1.1 the following claims are valid.*

(i) *For each prime ℓ the complex $R\Gamma_c(X_{\text{ét}}, f_*f^*\mathbb{Z}_\ell)$ belongs to $D^{\text{perf}}(\mathbb{Z}_\ell[G])$.*
(ii) *$R\Gamma(Y'_{W\text{ét}}, j_{Y,!}\mathbb{Z})$ belongs to $D^{\text{perf}}(\mathbb{Z}[G])$.*
(iii) *The sequence $\epsilon^{\text{Bock}}_{\mathbb{Z}_\ell,G,\phi}$ is exact.*

Proof. Claim (i) follows directly from Proposition 4.2(i) and (ii) with both $\mathcal{L} = \mathbb{Z}_\ell$ and $O = \mathbb{Z}_\ell$.

Next we note that the validity of condition (i) in Theorem 1.1 implies that the cohomology groups of the complex $R\Gamma(Y'_{W\text{ét}}, j_{Y,!}\mathbb{Z})$ are finitely generated in all degrees and vanish in almost all degrees. By the criterion of [18, Rapport, Lem. 4.5.1], it therefore follows that $R\Gamma(Y'_{W\text{ét}}, j_{Y,!}\mathbb{Z})$ belongs to $D^{\text{perf}}(\mathbb{Z}[G])$ if and only if it has finite Tor-dimension. Since Lemma 6.1(ii) implies that for every prime ℓ there is an isomorphism $\mathbb{Z}_\ell\otimes_{\mathbb{Z}} R\Gamma(Y'_{W\text{ét}}, j_{Y,!}\mathbb{Z}) \cong R\Gamma_c(X_{\text{ét}}, f_*f^*\mathbb{Z}_\ell)$ in $D(\mathbb{Z}_\ell[G])$, one can therefore derive claim (ii) as a straightforward consequence of claim (i).

For each prime ℓ we now write θ_ℓ for the element of the group $H^1(\text{Spec}(\mathbb{F}_q)_{\text{ét}}, \mathbb{Z}_\ell) = \text{Hom}_{\text{cont}}(\text{Gal}(\mathbb{F}_q^c/\mathbb{F}_q), \mathbb{Z}_\ell)$ that sends the Frobenius automorphism to 1 (and hence sends ϕ

to -1). We also write $\theta_{X,\ell}$ for the pullback of θ_ℓ to $H^1(X_{\text{ét}}, \mathbb{Z}_\ell)$ and recall (from Remarks 3.3 and 4.2) that $\epsilon^{\text{Bock}}_{\mathbb{Z}_\ell,G,\phi}$ coincides with the sequence of $\mathbb{Q}_\ell[G]$-modules that is obtained by scalar extension from the complex of $\mathbb{Z}_\ell[G]$-modules

$$0 \to H^0_c(X_{\text{ét}}, f_*f^*\mathbb{Z}_\ell) \xrightarrow{\cup\theta_{X,\ell}} H^1_c(X_{\text{ét}}, f_*f^*\mathbb{Z}_\ell) \xrightarrow{\cup\theta_{X,\ell}} H^2_c(X_{\text{ét}}, f_*f^*\mathbb{Z}_\ell) \xrightarrow{\cup\theta_{X,\ell}} \cdots$$

in which each differential is given by cup product with $\theta_{X,\ell}$. Claim (iii) is therefore true since the assumed validity of condition (ii) in Theorem 1.1 combines with the isomorphisms in Lemma 6.1(ii) to imply that this complex has finite cohomology groups. □

6.1.2. We are now ready to prove Theorem 1.1, the hypotheses of which we assume.

Then, from Lemma 6.2(ii), we know $R\Gamma(Y'_{W\text{ét}}, j_{Y,!}\mathbb{Z})$ belongs to $D^{\text{perf}}(\mathbb{Z}[G])$. In addition, the commutative diagram (16) combines with the direct sum decomposition (14) to imply that the claimed equality (3) is valid if and only if, for every prime ℓ, there is in $K_0(\mathbb{Z}_\ell[G], \mathbb{Q}_\ell[G])$ an equality

$$\begin{aligned}\delta_{\mathbb{Z}_\ell[G]}(\iota_\ell(Z^*(f,1))) &= -\pi_\ell(\chi_G(R\Gamma(Y'_{W\text{ét}}, j_{Y,!}\mathbb{Z}), \epsilon_{f,j}))\\ &= -\chi_{\mathbb{Z}_\ell[G]}(\mathbb{Z}_\ell \otimes_{\mathbb{Z}} R\Gamma(Y'_{W\text{ét}}, j_{Y,!}\mathbb{Z}), \mathbb{Q}_\ell \otimes_{\mathbb{Q}} \epsilon_{f,j})\\ &= -\chi_{\mathbb{Z}_\ell[G]}(R\Gamma_c(X_{\text{ét}}, f_*f^*\mathbb{Z}_\ell), \epsilon^{\text{Bock}}_{\mathbb{Z}_\ell,G,\phi}).\end{aligned}$$

Note that the validity of the second equality here is obvious and that the third equality is valid because the isomorphism in Lemma 6.1(ii) induces an identification of $\mathbb{Q}_\ell \otimes_{\mathbb{Q}} \epsilon_{f,j}$ with the (exact) sequence $\epsilon^{\text{Bock}}_{\mathbb{Z}_\ell,G,\phi}$, as already observed in the proof of Lemma 6.2(iii).

To complete the proof of Theorem 1.1 it is therefore enough to derive the above equality as a consequence of Theorem 4.3 in the case that $\mathcal{L} = \mathbb{Z}_\ell$ and $O = \mathbb{Z}_\ell$.

In fact, since the exactness of $\epsilon^{\text{Bock}}_{\mathbb{Z}_\ell,G,\phi}$ implies that the necessary 'semi-simplicity' hypothesis of Theorem 4.3 is satisfied (under the conditions of Theorem 1.1), this will clearly be the case provided that in $\zeta(\mathbb{Q}_\ell[[t]][G])$ one has an equality of series

$$\iota_\ell(Z(f,t)) = Z_G(X, \mathbb{Z}_\ell, t). \tag{34}$$

But, in terms of the ℓ-adic representations that are fixed at the beginning of §4.2.2, the series $Z_G(X, \mathbb{Z}_\ell, t)$ is defined (in (23)) as an Euler product over all x in X^0 of the terms

$$\begin{aligned}&\text{Nrd}_{\mathbb{Q}_\ell[G]}(1 - \phi^{\text{d}(x)}t^{\text{d}(x)} \mid \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} (\mathbb{Z}_\ell)_{G,\overline{x}})^{-1}\\ =& \sum_{\psi\in\text{Ir}_\ell(G)} \det_E(1 - \phi^{\text{d}(x)}t^{\text{d}(x)} \mid \text{Hom}_{E[G]}(V_{\check{\psi}}, E[G]))^{-1}e_\psi\\ =& \sum_{\psi\in\text{Ir}_\ell(G)} \det_E(1 - \phi^{\text{d}(x)}t^{\text{d}(x)} \mid V_\psi)^{-1}e_\psi.\end{aligned}$$

In view of this expansion, and the explicit definition of $Z(f,t)$ that is given in the Introduction, the required equality (34) therefore follows directly from the definition of the Artin L-series $L^{\text{Artin}}(Y, \chi, t)$ for each irreducible complex character χ of G, as given by Milne in [33, Exam. 13.6(b)].

This completes the proof of Theorem 1.1.

6.2. **Global Euler characteristics of Weil-étale cohomology.** In this section we derive Corollary 1.4 as a consequence of Theorem 1.1. Throughout we use the notation of §1.2.1.

6.2.1. We first discuss some useful preliminaries and, in particular, give a convenient description of the homomorphism h_G that occurs in Corollary 1.4.

To do this we use the fact that the long exact sequences of relative K-theory (13) with the pair $(\mathcal{A}, E)$ equal to $(\mathbb{Z}[G], \mathbb{Q})$ and to $(\mathbb{Z}_\ell[G], \mathbb{Q}_\ell)$ for all primes ℓ combine to give an exact commutative diagram

$$\begin{array}{ccccccc} \prod_\ell K_1(\mathbb{Z}_\ell[G]) & \longrightarrow & \prod'_\ell K_1(\mathbb{Q}_\ell[G]) & \xrightarrow{(\partial_{G,\ell})_\ell} & \bigoplus_\ell K_0(\mathbb{Z}_\ell[G], \mathbb{Q}_\ell[G]) & & \\ \uparrow & & \uparrow & & \uparrow \pi & & \\ K_1(\mathbb{Z}[G]) & \longrightarrow & K_1(\mathbb{Q}[G]) & \xrightarrow{\partial_G} & K_0(\mathbb{Z}[G], \mathbb{Q}[G]) & \xrightarrow{\partial'_G} & \mathrm{Cl}(\mathbb{Z}[G]). \end{array} \tag{35}$$

Here $\prod'_\ell K_1(\mathbb{Q}_\ell[G])$ denotes the restricted direct product of the groups $K_1(\mathbb{Q}_\ell[G])$ with respect to the subgroups given by the images of the natural maps $K_1(\mathbb{Z}_\ell[G]) \to K_1(\mathbb{Q}_\ell[G])$, the first two vertical arrows are the natural diagonal maps and π is the isomorphism $(\pi_\ell)_\ell$ from (14).

In addition, the commutative diagram (16) induces a commutative diagram

$$\begin{array}{ccc} \zeta(\mathbb{Q}[G])^\times & \xrightarrow{\delta_G} & K_0(\mathbb{Z}[G], \mathbb{Q}[G]) \\ \downarrow & & \downarrow \pi \\ \bigoplus_\ell \frac{\zeta(\mathbb{Q}_\ell[G])^\times}{\mathrm{Nrd}_{\mathbb{Q}_\ell[G]}(K_1(\mathbb{Z}_\ell[G]))} & \xrightarrow{(\delta_{\mathbb{Z}_\ell[G]})_\ell} & \bigoplus_\ell K_0(\mathbb{Z}_\ell[G], \mathbb{Q}_\ell[G]), \end{array} \tag{36}$$

in which the lower row is bijective. This diagram combines with the exactness of the lower row in (35) (and an application of the weak approximation theorem) to induce a canonical isomorphism

$$h'_G : \frac{J(\zeta(\mathbb{Q}[G]))}{\mathrm{im}(\Delta_G) \cdot U(\zeta(\mathbb{Q}[G]))} \cong \mathrm{Cl}(\mathbb{Z}[G]).$$

Here we write $J(\zeta(\mathbb{Q}[G]))$ for the idele group of the $\mathbb{Q}$-algebra $\zeta(\mathbb{Q}[G])$, $U(\zeta(\mathbb{Q}[G]))$ for the subgroup $\mathrm{im}(\mathrm{Nrd}_{\mathbb{R}[G]}) \cdot \prod_\ell \mathrm{Nrd}_{\mathbb{Q}_\ell[G]}(K_1(\mathbb{Z}_\ell[G]))$ of $J(\zeta(\mathbb{Q}[G]))$ and Δ_G for the diagonal embedding of $\zeta(\mathbb{Q}[G])^\times$ into $J(\zeta(\mathbb{Q}[G]))$.

Each element of the idele group $J(\zeta(\mathbb{Q}^c[G]))$ can be written uniquely as $x = \sum_{\chi \in \mathrm{Ir}(G)} x_\chi e_\chi$ with every x_χ in $J(\mathbb{Q}^c)$. With respect to this decomposition, an element x of $\zeta(\mathbb{Q}^c[G])^\times$, respectively $J(\zeta(\mathbb{Q}^c[G]))$, belongs to the subgroup $\zeta(\mathbb{Q}[G])^\times$, respectively $J(\zeta(\mathbb{Q}[G]))$, if and only if one has $\omega(x_\chi) = x_{\omega\circ\chi}$ for all ω in $\mathrm{Gal}(\mathbb{Q}^c/\mathbb{Q})$ and all χ in $\mathrm{Ir}(G)$.

In particular, the map from $\mathrm{Map}(\mathrm{Ir}(G), J(\mathbb{Q}^c))$ to $J(\zeta(\mathbb{Q}^c[G]))$ that sends each θ to $\sum_{\chi \in \mathrm{Ir}(G)} \theta(\chi) e_\chi$ restricts to give a homomorphism $\mathrm{Map}_{\mathbb{Q}}(\mathrm{Ir}(G), J(\mathbb{Q}^c)) \to J(\zeta(\mathbb{Q}[G]))$. The map h_G that occurs in Corollary 1.4 is then obtained by composing this latter homomorphism with h'_G.

We next fix a finite Galois extension N of $\mathbb{Q}$ in $\mathbb{Q}^c$ over which all characters in $\mathrm{Ir}(G)$ can be realised and write j_∞ for the natural embedding of $N^\times$ into $(N \otimes_{\mathbb{Q}} \mathbb{R})^\times \subset J(N)$. We also write $\mathrm{Ir}^{\mathrm{s}}(G)$ for the subset of $\mathrm{Ir}(G)$ comprising symplectic characters.

Then, for each x in $\zeta(\mathbb{Q}[G])^\times$ one has $x_\chi \in N^\times$ for all $\chi \in \mathrm{Ir}(G)$ and the Hasse-Schilling-Maass Norm Theorem (from, for example, [17, Th. (7.48)]) implies x belongs to $\mathrm{im}(\mathrm{Nrd}_{\mathbb{Q}[G]})$

if and only if for each χ in $\mathrm{Ir}^{\mathrm{s}}(G)$ every component of

$$j_\infty(x_\chi) \in (N \otimes_{\mathbb{Q}} \mathbb{R})^\times \subseteq (N \otimes_{\mathbb{Q}} \mathbb{C})^\times = \prod_{N\to\mathbb{C}} \mathbb{C}^\times$$

is a positive real number. Finally, for any element x of $\zeta(\mathbb{Q}[G])^\times$ we write x^{s} for the element of $\zeta(\mathbb{Q}[G])^\times$ that is specified by setting

$$(x^{\mathrm{s}})_\chi := \begin{cases} x_\chi, & \text{if } \chi \in \mathrm{Ir}^{\mathrm{s}}(G) \\ 1, & \text{if } \chi \in \mathrm{Ir}(G) \setminus \mathrm{Ir}^{\mathrm{s}}(G). \end{cases}$$

6.2.2. We are now ready to prove Corollary 1.4.

At the outset we recall that the leading term $Z^*(f,1)$ belongs to $\zeta(\mathbb{Q}[G])^\times$ (this is well-known but can also be derived as a consequence of the fact that for every prime ℓ one has $Z^*(f,1) = Z^*_G(X,\mathbb{Z}_\ell,1)$, as shown by the argument of §6.1.2).

It follows that the field N fixed in §6.2.1 contains $L^{\mathrm{Artin},*}(Y,\chi,1)$ for every χ in $\mathrm{Ir}(G)$ and that the function $z(f)$ on $\mathrm{Ir}(G)$ that sends each χ to $j_\infty(L^{\mathrm{Artin},*}(Y,\chi,1))$ belongs to $\mathrm{Map}_{\mathbb{Q}}(\mathrm{Ir}(G), J(N))$.

For each archimedean place v of $\mathbb{Q}^c$, we can then fix a choice of embedding $j_v : \mathbb{Q}^c \to \mathbb{C}$ so that the definition of $z^{\mathrm{Artin}}(f)$ in (4) implies that for every χ in $\mathrm{Ir}(G)$ and every place v of $\mathbb{Q}^c$ one has

$$z^{\mathrm{Artin}}(f)(\chi)_v = \begin{cases} z(f)(\chi)_{\tilde{v}}, & \text{if } \chi \in \mathrm{Ir}^{\mathrm{s}}(G) \text{ and } v \text{ is archimedean,} \\ 1, & \text{otherwise,} \end{cases}$$

where $\tilde{v}$ is the restriction of v to N. In particular, since the set $\mathrm{Ir}^{\mathrm{s}}(G)$ is stable under the action of $\mathrm{Gal}(\mathbb{Q}^c/\mathbb{Q})$, it follows that the function $z^{\mathrm{Artin}}(f)$ also belongs to $\mathrm{Map}_{\mathbb{Q}}(\mathrm{Ir}(G), J(\mathbb{Q}^c))$, as claimed in Corollary 1.4.

To proceed we write i_∞ for the natural inclusion of $\zeta(\mathbb{Q}[G])^\times$ into $\zeta(\mathbb{R}[G])^\times$ and $[x,y]$ for the element of the domain of the function h'_G that is represented by an element (x,y) of $J(\zeta(\mathbb{Q}[G]))$ where x belongs to $\zeta(\mathbb{R}[G])^\times$ and y to $\prod_\ell \zeta(\mathbb{Q}_\ell[G])^\times$.

Then, setting $C := R\Gamma(Y'_{W\text{ét}}, j_{Y,!}\mathbb{Z})$, the equality (3) implies that

$$\begin{aligned}
\chi_G(C) &= \partial'_G(\chi_G(C,\epsilon_{f,j})) \\
&= -\,\partial'_G(\delta_G(Z^*(f,1))) \\
&= -\,h'_G([1,(\iota_\ell(Z^*(f,1)))_\ell]) \\
&= h'_G([\iota_\infty(Z^*(f,1)),1]) \\
&= h'_G([\iota_\infty(Z^*(f,1)^{\mathrm{s}}),1]) \\
&= h_G(z^{\mathrm{Artin}}(f)),
\end{aligned} \tag{37}$$

as claimed in Corollary 1.4. Note that the third equality here follows from the commutativity of (36), the fourth is valid because $[\iota_\infty(Z^*(f,1)),(\iota_\ell(Z^*(f,1)))_\ell] = \Delta_G(Z^*(f,1))$ belongs to $\ker(h'_G)$, the fifth because $\iota_\infty(Z^*(f,1))/\iota_\infty(Z^*(f,1)^{\mathrm{s}})$ belongs to $\mathrm{im}(\mathrm{Nrd}_{\mathbb{R}[G]})$ and the last follows directly from the definition of $z^{\mathrm{Artin}}(f)$.

Now if x belongs to $\zeta(\mathbb{Q}[G])^\times$, then $x_\chi \in \mathbb{R}^\times$ for every χ in $\mathrm{Ir}^{\mathrm{s}}(G)$. In addition, since we are regarding $\mathbb{Q}^c$ as a subfield of $\mathbb{C}$, for each archimedean place v of $\mathbb{Q}^c$ the embedding j_v

is an automorphism of $\mathbb{Q}^c$ and so for every χ in $\mathrm{Ir}(G)$ one has both $j_v(x_\chi) = x_{j_v\circ\chi}$ and also $j_v \circ \chi \in \mathrm{Ir}^{\mathrm{s}}(G)$ if and only if $\chi \in \mathrm{Ir}^{\mathrm{s}}(G)$.

In particular, if x is any element of $\zeta(\mathbb{Q}[G])^\times$ such that for every χ in $\mathrm{Ir}^{\mathrm{s}}(G)$ the real numbers x_χ and $L^{\mathrm{Artin},*}(Y,\chi,1)$ have the same sign, then the elements $[\iota_\infty(x)^{\mathrm{s}},1]$ and $[\iota_\infty(Z^*(f,1)^{\mathrm{s}}),1]$ coincide.

Taken in conjunction with the equality (37) this shows that $\chi_G(C)$ depends only on the signs of $L^{\mathrm{Artin},*}(Y,\chi,1)$ for all χ in $\mathrm{Ir}^{\mathrm{s}}(G)$, as claimed in Corollary 1.4.

In a similar way, one deduces that the order of $\chi_G(C)$ divides 2 since $\iota_\infty(Z^*(f,1)^{\mathrm{s}})^2$ belongs to $\mathrm{im}(\mathrm{Nrd}_{\mathbb{R}[G]})$ and that $[\iota_\infty(Z^*(f,1)^{\mathrm{s}}),1]$, and hence also $\chi_G(C)$, vanishes whenever $L^{\mathrm{Artin},*}(Y,\chi,1)$ is positive for every χ in $\mathrm{Ir}^{\mathrm{s}}(G)$.

Finally, we note that since C belongs to $D^{\mathrm{perf}}(\mathbb{Z}[G])$ and is acyclic in all negative degrees, a standard resolution argument shows that there exists a natural number t such that C is isomorphic in $D(\mathbb{Z}[G])$ to a bounded complex of finitely generated projective G-modules

$$P_{-1} \to P_0 \to P_1 \to \cdots \to P_t$$

in which P_i is free of rank r_i for each $i \geq 0$. By Swan's Theorem [17, (32.1)], the $\mathbb{Q}[G]$-module spanned by P_{-1} is free and, for convenience, one can fix the above complex so that the rank r of this module is greater than one.

In particular, if $\chi_G(C)$ vanishes, then one has $r = \sum_{i\geq 0}(-1)^i r_i$ and hence also in $K_0(\mathbb{Z}[G])$ an equality

$$[P_{-1}] = \sum\nolimits_{i\geq 0}(-1)^i[P_i] = \sum\nolimits_{i\geq 0}(-1)^i[\mathbb{Z}[G]^{r_i}] = [\mathbb{Z}[G]^r],$$

where we write $[M]$ for the class in $K_0(\mathbb{Z}[G])$ of a finitely generated projective $\mathbb{Z}[G]$-module M. Then, since $r > 1$, this equality combines with the Bass Cancellation Theorem [17, (41.20)] to imply that P_{-1} is a free G-module of rank r.

This shows that C is isomorphic in $D(\mathbb{Z}[G])$ to a bounded complex of finitely generated free G-modules whenever $\chi_G(C)$ vanishes and hence completes the proof of Corollary 1.4.

7. Affine curves

In this section we derive a range of concrete consequences of Theorem 1.1 in the special case that X is an affine curve and $\mathbb{F}_q$ is isomorphic to the field of constants of the function field of X. In particular, in this way we will prove Theorem 1.5 as well as Corollaries 1.6 and 1.7.

For the reader's convenience, we first fix some basic notation that will be used throughout this section.

For any global function field E we write C_E for the unique geometrically irreducible smooth projective curve that has function field E.

We fix a finite Galois extension of such fields F/k and set $G := \mathrm{Gal}(F/k)$. We also fix a finite non-empty set of places Σ of k that contains all places that ramify in F/k.

For each intermediate field E of F/k we write Σ_E for the set of places of E that lie above those in Σ and $\mathcal{O}_{E,\Sigma}$ for the subring of E comprising elements that are integral at all places outside Σ_E. We then write C_E^Σ for the affine curve $\mathrm{Spec}(\mathcal{O}_{E,\Sigma})$, $j_E^\Sigma : C_E^\Sigma \to C_E$ for the corresponding open immersion and

$$f_\Sigma : C_F^\Sigma \to C_k^\Sigma \ \text{ and } \ f'_\Sigma : C_F \to C_k$$

for the morphisms that are induced by the inclusion of fields $k \subseteq F$.

We fix q so that $\mathbb{F}_q$ identifies with the constant field of k and regard all of the above morphisms as morphisms of $\mathbb{F}_q$-schemes.

We identify each place w of E with the corresponding (closed) point of C_E. For each such w we write ord_w for its normalised additive valuation (on E) and $\mathrm{d}(w)$ for its degree over $\mathbb{F}_q$. We also write $\kappa(w)$ for its residue field of w and set $Z_w := \mathrm{Spec}(\kappa(w))$.

We write $\mathcal{O}_{F,\Sigma}^\times$ for the unit group of $\mathcal{O}_{F,\Sigma}$, set $B_{F,\Sigma} := \bigoplus_{w\in\Sigma_F}\mathbb{Z}$ and write $B_{F,\Sigma}^0$ for the kernel of the homomorphism $B_{F,\Sigma} \to \mathbb{Z}$ that sends each element $(n_w)_w$ to $\sum_w n_w$. We note that each of the groups $\mathcal{O}_{F,\Sigma}, \mathcal{O}_{F,\Sigma}^\times$, $B_{F,\Sigma}$ and $B_{F,\Sigma}^0$ has a natural action of G.

By a slight abuse of notation, we denote by $\mathbb{G}_m$ the Weil-étale sheaf that is obtained by restricting the étale sheaf $\mathbb{G}_m$ to the Weil-étale site.

7.1. **The equivariant leading term conjecture.** In this section we give a precise statement of Theorem 1.5.

To do this we recall (from, for example, [2, Lem. 1]) that the complex

$$K_{F,\Sigma}^\bullet := R\Gamma(C_{F,\mathrm{W\acute{e}t}}^\Sigma, \mathbb{G}_m)$$

belongs to $D^{\mathrm{perf}}(\mathbb{Z}[G])$, is acyclic outside degrees zero and one and is such that the explicit computations of [5] induce canonical identifications

$$H^0(K_{F,\Sigma}^\bullet) = \mathcal{O}_{F,\Sigma}^\times, \ \ H^1(K_{F,\Sigma}^\bullet)_{\mathrm{tor}} = \mathrm{Cl}(\mathcal{O}_{F,\Sigma}) \ \text{ and } \ H^1(K_{F,\Sigma}^\bullet)_{\mathrm{tf}} = B_{F,\Sigma}^0. \tag{38}$$

We write

$$\mathrm{D}_{F,\Sigma}^q : \mathcal{O}_{F,\Sigma}^\times \to B_{F,\Sigma}^0 \tag{39}$$

for the homomorphism of G-modules that sends each u in $\mathcal{O}_{F,\Sigma}^\times$ to $(\mathrm{ord}_w(u)\cdot \mathrm{d}(w))_{w\in\Sigma_F}$. Then the induced map $\mathbb{Q}\otimes_\mathbb{Z}\mathrm{D}_{F,\Sigma}^q$ is bijective (by the Riemann-Roch Theorem) and hence combines with the identifications in (38) to give an exact sequence of $\mathbb{Q}[G]$-modules

$$\epsilon_{F,\Sigma} : 0 \to \mathbb{Q}\otimes_\mathbb{Z} H^0(K_{F,\Sigma}^\bullet) \xrightarrow{\mathbb{Q}\otimes_\mathbb{Z}\mathrm{D}_{F,\Sigma}^q} \mathbb{Q}\otimes_\mathbb{Z} H^1(K_{F,\Sigma}^\bullet) \xrightarrow{0} \mathbb{Q}\otimes_\mathbb{Z} H^2(K_{F,\Sigma}^\bullet) \xrightarrow{0} \cdots$$

Finally, we write $x \mapsto x^\#$ for the $\mathbb{Q}$-linear anti-involution of $\mathbb{Q}[G]$ that inverts elements of G.

We can now give a more explicit statement of Theorem 1.5.

Theorem 7.1. *In $K_0(\mathbb{Z}[G],\mathbb{Q}[G])$ one has*

$$\delta_G(Z^*(f_\Sigma,1)^\#) = \chi_G(R\Gamma(C_{F,\mathrm{W\acute{e}t}}^\Sigma,\mathbb{G}_m),\epsilon_{F,\Sigma}).$$

Remark 7.2. Let m be the degree of $\mathbb{F}_q$ over $\mathbb{F}_p$. Then $Z(f_\Sigma,t)$ and $\mathrm{D}_{F,\Sigma}^q$ are related to the function $Z_{F/k,\Sigma}(t)$ and homomorphism $D_{F,\Sigma}$ that occur in [2, Lem. 2] by the equalities $Z_{F/k,\Sigma}(t) = Z(f_\Sigma, t^m)$ and $\mathrm{D}_{F,\Sigma} = [m]\circ \mathrm{D}_{F,\Sigma}^q$ where $[m]$ denotes the endomorphism of $B_{F,\Sigma}^0$ given by multiplication by m. These equalities can be used to show that the equality in Theorem 7.1 is equivalent to that of [2, (3)] and hence that Theorem 7.1 combines with [2, Lem. 2] to imply the validity of the central conjecture (Conj. C(F/k)) of [2]. The computation of [2, Prop. 4.1] also then implies that Theorem 7.1 is equivalent to the function field case of [4, Conj. LTC(F/k)]. Finally, we recall that the Artin-Verdier Duality Theorem induces (via [32, Th. 5.4(a)]) a canonical isomorphism in $D^{\rm perf}(\mathbb{Z}[G])$

$$\Delta_\Sigma : R\Gamma(C^\Sigma_{F,\mathrm{W\acute{e}t}}, \mathbb{G}_m) \cong R\,\mathrm{Hom}_\mathbb{Z}(R\Gamma(C_{F,\mathrm{W\acute{e}t}}, j^\Sigma_{F,!}\mathbb{Z}), \mathbb{Z}[-2]), \tag{40}$$

where G acts contragrediently on the linear dual complex. This morphism Δ_Σ can be combined with the isomorphisms in Lemma 6.1(ii) to show Theorem 7.1 generalises the main result (Theorem 3.1) of [3].

7.2. **The proof of Theorem 7.1.** We write ψ^* for the involution of $K_0(\mathbb{Z}[G], \mathbb{Q}[G])$ that for each pair of finitely generated projective G-modules P and Q and isomorphism of $\mathbb{Q}[G]$-modules $\mu : \mathbb{Q}\otimes_\mathbb{Z} P \to \mathbb{Q}\otimes_\mathbb{Z} Q$ sends (P, μ, Q) to $(\mathrm{Hom}_\mathbb{Z}(P, \mathbb{Z}), \mathrm{Hom}_\mathbb{Q}(\mu, \mathbb{Q})^{-1}, \mathrm{Hom}_\mathbb{Z}(Q, \mathbb{Z}))$, where the linear duals are endowed with the contragredient action of G and so are projective $\mathbb{Z}[G]$-modules.

Then an easy exercise (using the commutativity of (16)) shows that for every element x of $\zeta(\mathbb{Q}[G])^\times$ one has

$$\psi^*(\delta_G(x)) = -\delta_G(x^\#). \tag{41}$$

Given this equality, the isomorphism $R\,\mathrm{Hom}_\mathbb{Z}(\Delta_\Sigma, \mathbb{Z}[-2])$ implies that Theorem 7.1 is valid if and only if in $K_0(\mathbb{Z}[G], \mathbb{Q}[G])$ one has

$$\delta_G(Z^*(f_\Sigma, 1)) = -\chi_G(R\Gamma(C_{F,\mathrm{W\acute{e}t}}, j^\Sigma_{F,!}\mathbb{Z}), \epsilon^*_{F,\Sigma}). \tag{42}$$

Here we write $\epsilon^*_{F,\Sigma}$ for the exact sequence of $\mathbb{Q}[G]$-modules that is obtained by taking the $\mathbb{Q}$-linear dual of $\epsilon_{F,\Sigma}$ and then using the maps $\mathrm{Hom}_\mathbb{Q}(\mathbb{Q}\otimes_\mathbb{Z} H^i(\Delta_\Sigma), \mathbb{Q})$ to identify $\mathrm{Hom}_\mathbb{Q}(\mathbb{Q}\otimes_\mathbb{Z} H^i(K^\bullet_{F,\Sigma}), \mathbb{Q})$ with $\mathbb{Q}\otimes_\mathbb{Z} H^{2-i}(C_{F,\mathrm{W\acute{e}t}}, j^\Sigma_{F,!}\mathbb{Z})$.

Our aim is to deduce the required equality (42) by applying Theorem 1.1 to the data

$$Y = C^\Sigma_F,\ Y' = C_F,\ X = C^\Sigma_k,\ X' = C_k,\ j = j_k := j^\Sigma_k, j_Y = j_F := j^\Sigma_F, f = f_\Sigma \text{ and } f' = f'_\Sigma.$$

In this regard we note that X is geometrically connected and, since Y' is a curve, that the conditions (i) and (ii) of Theorem 1.1 are already known to be satisfied (by [32, Th. 7.4]). Theorem 1.1 therefore applies in this case to give an equality

$$\delta_G(Z^*(f_\Sigma, 1)) = -\chi_G(R\Gamma(C_{F,\mathrm{W\acute{e}t}}, j_{F,!}\mathbb{Z}), \epsilon_{f_\Sigma, j_k}).$$

To derive (42) it is therefore enough to show that the exact sequences $\epsilon^*_{F,\Sigma}$ and ϵ_{f_Σ,j_k} coincide and this follows directly from the next result.

Lemma 7.3. *The following diagram of $\mathbb{Q}[G]$-modules commutes.*

$$\begin{array}{ccc}
\mathbb{Q}\otimes_\mathbb{Z} H^1(C_{F,\mathrm{W\acute{e}t}}, j^\Sigma_{F,!}\mathbb{Z}) & \xrightarrow{\mathrm{Hom}_\mathbb{Z}(H^1(\Delta_\Sigma),\mathbb{Q})} & \mathrm{Hom}_\mathbb{Z}(B^0_{F,\Sigma}, \mathbb{Q}) \\
{\scriptstyle \mathbb{Q}\otimes_\mathbb{Z}(\cup\theta_{C_F})}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Hom}_\mathbb{Z}(\mathrm{D}^q_{F,\Sigma},\mathbb{Q})} \\
\mathbb{Q}\otimes_\mathbb{Z} H^2(C_{F,\mathrm{W\acute{e}t}}, j^\Sigma_{F,!}\mathbb{Z}) & \xrightarrow{\mathrm{Hom}_\mathbb{Z}(H^0(\Delta_\Sigma),\mathbb{Q})} & \mathrm{Hom}_\mathbb{Z}(\mathcal{O}^\times_{F,\Sigma}, \mathbb{Q})
\end{array}$$

Proof. It is enough to prove that the diagram commutes after applying $\mathbb{Q}_\ell\otimes_\mathbb{Q} -$ for any prime ℓ. We therefore fix a prime ℓ and, following Lemma 6.1(ii), we consider $H^i(C_{k,\mathrm{\acute{e}t}}, j_!\mathcal{F}_\ell)$ with $j := j^\Sigma_k$ and $\mathcal{F}_\ell := f_*f^*\mathbb{Z}_\ell$ rather than $H^i(C_{F,\mathrm{W\acute{e}t}}, j^\Sigma_{F,!}\mathbb{Z})$.

We write Z^Σ for the complement of C_k^Σ in C_k and $i : Z^\Sigma \to C_k$ for the natural closed immersion. For each $y \in \Sigma$ we also write $i_y : Z_y \to C_k$ for the natural closed immersion and $S_y(F)$ for the set of places of F above y.

Now $\mathcal{F}_\ell = j^*\mathcal{F}'_\ell$ with $\mathcal{F}'_\ell := f'_*f'^*\mathbb{Z}_\ell$ so there exists a natural exact sequence of $\mathbb{Z}_\ell[G]$-sheaves $0 \to j_!\mathcal{F}_\ell \to \mathcal{F}'_\ell \to i_*i^*\mathcal{F}'_\ell \to 0$ on C_k and hence a composite morphism in $D(\mathbb{Z}_\ell[G])$

$$\xi : \bigoplus_{y\in\Sigma} R\Gamma(Z_{y,\text{ét}}, i_y^*\mathcal{F}'_\ell) \cong R\Gamma(C_{k,\text{ét}}, i_*i^*\mathcal{F}'_\ell) \to R\Gamma(C_{k,\text{ét}}, j_!\mathcal{F}_\ell)[1].$$

This morphism in turn induces a commutative diagram of $\mathbb{Z}_\ell[G]$-modules

$$\begin{array}{ccccc} \bigoplus_{y\in\Sigma} H^0(Z_{y,\text{ét}}, i_y^*\mathcal{F}'_\ell) & \xrightarrow{H^0(\xi)} & H^1(C_{k,\text{ét}}, j_!\mathcal{F}_\ell) & \xrightarrow{\mathrm{Hom}_{\mathbb{Z}}(H^1(\Delta_\Sigma),\mathbb{Q}_\ell)} & \mathrm{Hom}_{\mathbb{Z}}(B^0_{F,\Sigma}, \mathbb{Q}_\ell) \\ \Big\downarrow{\scriptstyle (-1)\times(\cup\theta_{Z_y,\ell})_y} & & \Big\downarrow{\scriptstyle \cup\theta_{C_k,\ell}} & & \\ \bigoplus_{y\in\Sigma} H^1(Z_{y,\text{ét}}, i_y^*\mathcal{F}'_\ell) & \xrightarrow{H^1(\xi)} & H^2(C_{k,\text{ét}}, j_!\mathcal{F}_\ell) & \xrightarrow{\mathrm{Hom}_{\mathbb{Z}}(H^0(\Delta_\Sigma),\mathbb{Q}_\ell)} & \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}^\times_{F,\Sigma}, \mathbb{Q}_\ell) \end{array} \tag{43}$$

where the factor -1 occurs in the left vertical homomorphism because ξ maps to the 1-shift of $R\Gamma(C_{k,\text{ét}}, j_!\mathcal{F}_\ell)$.

Now each complex $\mathcal{E}_y^\bullet := R\Gamma(Z_{y,\text{ét}}, i_y^*\mathcal{F}'_\ell)$ is canonically isomorphic to the direct sum over w in $S_y(F)$ of the complexes

$$\mathbb{Z}_\ell[G_w/G_w^0]^\# \xrightarrow{1-\sigma_w} \mathbb{Z}_\ell[G_w/G_w^0]^\#.$$

Here the first term occurs in degree 0, G_w and G_w^0 denote the decomposition and inertia groups of w in G, σ_w the Frobenius automorphism in $G_w/G_w^0 \cong \mathrm{Gal}(\kappa(w)/\kappa(v))$ and $\mathbb{Z}_\ell[G_w/G_w^0]^\#$ the left $\mathbb{Z}_\ell[G_w]$-module $\mathbb{Z}_\ell[G_w/G_w^0]$ upon which σ_w acts as right multiplication by σ_w^{-1}.

To compute the groups $H^i(\mathcal{E}_y^\bullet)$ explicitly we must use the conventions of [5] since they underlie the explicit descriptions of the cohomology of $K^\bullet_{F,\Sigma}$ that are given in (38). We therefore use the isomorphism $\mathcal{E}_y^\bullet \cong \bigoplus_{w\in S_y(F)} R\Gamma(Z_{y,\text{ét}}, \mathbb{Z}_\ell[G_w/G_w^0])$ in $D(\mathbb{Z}_\ell[G])$ that is induced by the morphisms of complexes

$$\begin{array}{ccc} \mathbb{Z}_\ell[G_w/G_w^0]^\# & \xrightarrow{1-\sigma_w} & \mathbb{Z}_\ell[G_w/G_w^0]^\# \\ \Big\| & & \Big\downarrow{\scriptstyle -\sigma_w} \\ \mathbb{Z}_\ell[G_w/G_w^0] & \xrightarrow{1-\sigma_w} & \mathbb{Z}_\ell[G_w/G_w^0]. \end{array} \tag{44}$$

We then identify $H^0(Z_{y,\text{ét}}, \mathbb{Z}_\ell[G_w/G_w^0])$ and $H^1(Z_{y,\text{ét}}, \mathbb{Z}_\ell[G_w/G_w^0])$ with $H^0(Z_{w,\text{ét}}, \mathbb{Z}_\ell) \cong \mathbb{Z}_\ell$ and $H^1(Z_{w,\text{ét}}, \mathbb{Z}_\ell) \cong \mathrm{Hom}_{\mathrm{cont}}(\mathrm{Gal}(\kappa(w)^c/\kappa(w)), \mathbb{Z}_\ell) \cong \mathbb{Z}_\ell$ in the natural way, where the last map evaluates each homomorphism at the topological generator $\phi^{-\mathrm{d}(w)}$ of $\mathrm{Gal}(\kappa(w)^c/\kappa(w))$.

By passing to cohomology in diagram (44), we therefore obtain a commutative diagram

$$\begin{array}{ccccc} \bigoplus_{y\in\Sigma}(\bigoplus_{w\in S_y(F)}\mathbb{Q}_\ell) & \xrightarrow{\cong} & \bigoplus_{y\in\Sigma}(\mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} H^0(\mathcal{E}_y^\bullet)) & \xrightarrow{a_0} & \mathrm{Hom}_{\mathbb{Z}}(B^0_{F,\Sigma}, \mathbb{Q}_\ell) \\ {\scriptstyle a_2}\Big\downarrow & & \Big\downarrow{\scriptstyle (-1)\times(\cup\theta_{Z_y,\ell})_y} & & \\ \bigoplus_{y\in\Sigma}(\bigoplus_{w\in S_y(F)}\mathbb{Q}_\ell) & \xrightarrow{\cong} & \bigoplus_{y\in\Sigma}(\mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} H^1(\mathcal{E}_y^\bullet)) & \xrightarrow{a_1} & \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}^\times_{F,\Sigma}, \mathbb{Q}_\ell) \end{array} \tag{45}$$

in which a_0 and a_1 are the homomorphisms induced by the upper and lower rows of (43) respectively and a_2 is defined to make the first square commute. Now the composite upper, resp. lower, horizontal map in (45) is equal to $\mathrm{Hom}_{\mathbb{Z}}(a_3, \mathbb{Q}_\ell)^{-1}$ with a_3 equal to the natural map $B^0_{F,\Sigma} \subset B_{F,\Sigma} \cong \bigoplus_{y\in\Sigma}(\bigoplus_{w\in S_y(F)}\mathbb{Z})$, resp. with a_3 equal to the map $\mathcal{O}^\times_{F,\Sigma} \to \bigoplus_{y\in\Sigma}(\bigoplus_{w\in S_y(F)}\mathbb{Z})$ sending u to $(-\mathrm{ord}_w(u))_w$ where the minus sign occurs because the morphism (44) induces $-\mathrm{id}_{\mathbb{Z}_\ell}$ on cohomology in degree 1.

In particular, since the upper horizontal map in the diagram (45) is surjective the claimed commutativity of the stated diagram will follow from (43) and (45) provided that a_2 sends each element $(x_w)_w$ to $(-\mathrm{d}(w)x_w)_w$, and this is true since for each w in $S_y(F)$ the map $(-1)\times\theta_{Z_y,\ell}$ sends $\phi^{-\mathrm{d}(w)}$ to

$$-\theta_{Z_y,\ell}(\phi^{-\mathrm{d}(w)}) = -(-\mathrm{d}(w))\times\theta(\phi) = \mathrm{d}(w)\times(-1) = -\mathrm{d}(w).$$

□

7.3. **Weil-étale cohomology and Selmer modules.** In order to relate Theorem 7.1 to several previously formulated refinements of Stark's Conjecture, it is convenient to first clarify the link between Weil-étale cohomology and the canonical 'Selmer modules' of $\mathbb{G}_m$ that are introduced by Kurihara, Sano and the first author in [8]. In this section we shall describe this link precisely and also derive some interesting consequences that it has for the Galois structure of Selmer groups.

The Pontryagin dual $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ of a G-module M is written as $M^\vee$ and regarded as endowed with the contragredient action of G.

For a complex X in $D^-(\mathbb{Z}[G])$ we set $X^* := R\,\mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z})$ and $X^\vee := R\,\mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Q}/\mathbb{Z})$ and regard both as objects of $D(\mathbb{Z}[G])$ via the contragredient action of G. We use the fact that the contravariant functor $X \mapsto X^*$ preserves the subcategory $D^{\mathrm{perf}}(\mathbb{Z}[G])$ of $D^-(\mathbb{Z}[G])$ and is also self-inverse as an endofunctor of $D^{\mathrm{perf}}(\mathbb{Z}[G])$.

7.3.1. For any finite disjoint sets of places Σ_1 and Σ_2 of k, the Σ_1-relative, Σ_2-trivialized, Selmer module $\mathrm{Sel}^{\Sigma_2}_{\Sigma_1}(F)$ for $\mathbb{G}_m$ over F is defined in [8, §2] to be the cokernel of the homomorphism

$$\prod_{w\notin(\Sigma_1\cup\Sigma_2)_F}\mathbb{Z} \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(F^\times_{\Sigma_2}, \mathbb{Z}). \tag{46}$$

Here $F^\times_{\Sigma_2}$ denotes the subgroup $\{a \in F^\times : \mathrm{ord}_w(a-1) > 0 \text{ for all } w \in \Sigma_{2,F}\}$ of $F^\times$ and the arrow sends $(x_w)_w$ to the homomorphism $a \mapsto \sum_{w\notin(\Sigma_1\cup\Sigma_2)_F}\mathrm{ord}_w(a)\cdot x_w$.

We further recall from [8, Prop. 2.2] that if Σ_1 is non-empty, then there exists a canonical exact sequence

$$0 \to \mathrm{Cl}^{\Sigma_2}_{\Sigma_1}(F)^\vee \to \mathrm{Sel}^{\Sigma_2}_{\Sigma_1}(F) \to \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}^\times_{F,\Sigma_1,\Sigma_2}, \mathbb{Z}) \to 0. \tag{47}$$

Here we write $\mathcal{O}^\times_{F,\Sigma_1,\Sigma_2}$ for the finite index subgroup $\mathcal{O}^\times_{F,\Sigma_1}\cap F^\times_{\Sigma_2}$ of $\mathcal{O}^\times_{F,\Sigma_1}$ and $\mathrm{Cl}^{\Sigma_2}_{\Sigma_1}(F)$ for the ray class group of $\mathcal{O}_{F,\Sigma_1}$ modulo the product of all places in $\Sigma_{2,F}$.

7.3.2. If w is any place of F outside Σ_F, then the closed immersion $i_w : Z_w \to C_F^\Sigma$ induces a morphism of sheaves $\mathbb{G}_m \to i_{w,*}i_w^*\mathbb{G}_m$ on $C^\Sigma_{F,\mathrm{W\acute{e}t}}$ and hence a morphism in $D(\mathbb{Z}[G])$

$$\varrho_w : R\Gamma(C^\Sigma_{F,\mathrm{W\acute{e}t}}, \mathbb{G}_m) \to R\Gamma(C^\Sigma_{F,\mathrm{W\acute{e}t}}, i_{w,*}i_w^*\mathbb{G}_m) \cong R\Gamma(\kappa(w)_{\mathrm{W\acute{e}t}}, \mathbb{G}_m).$$

If Σ' is any finite non-empty set of places of k that is disjoint from Σ, then we define the Σ'-modified Weil-étale cohomology complex $R\Gamma_{\Sigma'}(C^\Sigma_{F,\mathrm{W\acute{e}t}}, \mathbb{G}_m)$ so that it lies in an exact triangle in $D(\mathbb{Z}[G])$ of the form

$$R\Gamma_{\Sigma'}(C^\Sigma_{F,\mathrm{W\acute{e}t}}, \mathbb{G}_m) \to R\Gamma(C^\Sigma_{F,\mathrm{W\acute{e}t}}, \mathbb{G}_m) \xrightarrow{\varrho^{\Sigma'}_\Sigma} \bigoplus_{w\in\Sigma'_F} R\Gamma(\kappa(w)_{\mathrm{W\acute{e}t}}, \mathbb{G}_m) \to \cdot \tag{48}$$

where $\varrho^{\Sigma'}_\Sigma$ denotes the diagonal morphism $(\varrho_w)_{w\in\Sigma'_F}$.

Since each place v in Σ' is unramified in F we can fix a place w_v of F above v and write σ_v for the Frobenius element of w_v in G. We then define a Σ'-modified leading term element in $\zeta(\mathbb{Q}[G])^\times$ by setting

$$Z^*_{\Sigma'}(f_\Sigma, 1) = Z^*(f_\Sigma, 1) \cdot \prod_{v\in\Sigma'} \mathrm{Nrd}_{\mathbb{Q}[G]}([1 - \mathrm{N}v \cdot \sigma_v]_r)$$

where $[1-\mathrm{N}v\cdot\sigma_v]_r$ denotes the element of $K_1(\mathbb{Q}[G])$ that corresponds to the automorphism of $\mathbb{Q}[G]$ given by right multiplication by $1 - \mathrm{N}v \cdot \sigma_v$.

In the following result we record relevant properties of these constructions and also use them to reinterpret Theorem 7.1.

Proposition 7.4. *Set* $K^{\Sigma',\bullet}_{F,\Sigma} := R\Gamma_{\Sigma'}(C^\Sigma_{F,\mathrm{W\acute{e}t}}, \mathbb{G}_m)$. *Then the following claims are valid.*

(i) *The complexes* $K^{\Sigma',\bullet}_{F,\Sigma}$ *and* $(K^{\Sigma',\bullet}_{F,\Sigma})^*[-1]$ *belong to* $D^{\mathrm{perf}}(\mathbb{Z}[G])$ *and are acyclic outside degrees zero and one.*

(ii) $H^1((K^{\Sigma',\bullet}_{F,\Sigma})^*)$ *is the cokernel of the diagonal map* $\mathbb{Z} \to B^*_{F,\Sigma}$ *and* $H^2((K^{\Sigma',\bullet}_{F,\Sigma})^*)$ *identifies with* $\mathrm{Sel}^{\Sigma'}_\Sigma(F)$.

(iii) $H^0(K^{\Sigma',\bullet}_{F,\Sigma}) = \mathcal{O}^\times_{F,\Sigma,\Sigma'}$, $H^1(K^{\Sigma',\bullet}_{F,\Sigma})_{\mathrm{tor}} = \mathrm{Cl}^{\Sigma'}_\Sigma(F)$ *and* $H^1(K^{\Sigma',\bullet}_{F,\Sigma})_{\mathrm{tf}} = B^0_{F,\Sigma}$.

(iv) *In* $K_0(\mathbb{Z}[G], \mathbb{Q}[G])$ *there are equalities*

$$\delta_G(Z^*_{\Sigma'}(f_\Sigma, 1)^\#) = \chi_G(K^{\Sigma',\bullet}_{F,\Sigma}, \epsilon_{F,\Sigma}) \quad \textit{and} \quad \delta_G(Z^*_{\Sigma'}(f_\Sigma, 1)) = \chi_G((K^{\Sigma',\bullet}_{F,\Sigma})^*[-1], \epsilon^*_{F,\Sigma}).$$

Proof. We set $\mathbb{F}^\times_{\Sigma'} := \bigoplus_{w\in\Sigma'_F} \kappa(w)^\times$ and $K^\bullet_{\Sigma'} := \bigoplus_{w\in\Sigma'_F} R\Gamma(\kappa(w)_{\mathrm{W\acute{e}t}}, \mathbb{G}_m)$.

Then, since the complex $R\Gamma_c((\mathcal{O}_{F,\Sigma})_{\mathcal{W}}, \mathbb{Z})$ constructed in [8, Prop. 2.4] is canonically isomorphic to $R\Gamma(C_{F,\mathrm{W\acute{e}t}}, j^\Sigma_{F,!}\mathbb{Z})$ (cf. [8, Rem. 2.5(i)]), there is a diagram in $D(\mathbb{Z}[G])$

$$\begin{array}{ccccccc}
(\mathbb{F}^\times_{\Sigma'})^\vee[-3] & \xrightarrow{\theta''_{\Sigma,\Sigma'}} & R\Gamma_c((\mathcal{O}_{F,\Sigma})_{\mathcal{W}}, \mathbb{Z}) & \longrightarrow & R\Gamma_{c,\Sigma'}((\mathcal{O}_{F,\Sigma})_{\mathcal{W}}, \mathbb{Z}) & \longrightarrow & \\
\downarrow{\scriptstyle\iota} & & \downarrow{\scriptstyle(\Delta_\Sigma)^*[-2]} & & & & \\
(K^\bullet_{\Sigma'})^*[-2] & \xrightarrow{(\varrho^{\Sigma'}_\Sigma)^*} & R\Gamma(C^\Sigma_{F,\mathrm{W\acute{e}t}}, \mathbb{G}_m)^*[-2] & \longrightarrow & (K^{\Sigma',\bullet}_{F,\Sigma})^*[-2] & \longrightarrow & .
\end{array} \tag{49}$$

Here the upper row is the exact triangle obtained by shifting the exact triangle in the third column of [8, (6)] whilst the lower row is the shift of the defining exact triangle (48). The morphism Δ_Σ is the duality isomorphism (40) and ι is the canonical isomorphism $\big(\mathbb{F}_{\Sigma'}^\times\big)^\vee[-3] \cong (K_{\Sigma'}^\bullet)^*[-2]$ in $D(\mathbb{Z}[G])$ that is induced by the fact that for each w in Σ'_F the complex $R\Gamma(\kappa(w)_{\mathrm{W\acute{e}t}}, \mathbb{G}_m)$ is canonically isomorphic to $\kappa(w)^\times[0]$.

Assuming for the moment that the above diagram is commutative, it can be completed to give an isomorphism in $D(\mathbb{Z}[G])$ from $(K_{F,\Sigma}^{\Sigma',\bullet})^*[-1]$ to $R\Gamma_{c,\Sigma'}((\mathcal{O}_{F,\Sigma})_{\mathcal{W}}, \mathbb{Z})[1]$.

Given this isomorphism, both claim (ii) and the assertions in claim (i) regarding the complex $(K_{F,\Sigma}^{\Sigma',\bullet})^*[-1]$ follow directly from those of [8, Prop. 2.4(iii) and (iv)] and the fact $F_{\Sigma'}^\times$ is torsion-free since Σ' is non-empty and F has characteristic p.

From the isomorphism $K_{F,\Sigma}^{\Sigma',\bullet} \cong ((K_{F,\Sigma}^{\Sigma',\bullet})^*[-1])^*[-1]$ we can then deduce $K_{F,\Sigma}^{\Sigma',\bullet}$ belongs to $D^{\mathrm{perf}}(\mathbb{Z}[G])$ and also obtain (via the universal coefficient spectral sequence) a description of the cohomology of $K_{F,\Sigma}^{\Sigma',\bullet}$ in terms of that of $(K_{F,\Sigma}^{\Sigma',\bullet})^*[-1]$. In this way, claim (ii) and the exact sequence (47) with $\Sigma_1 = \Sigma$ and $\Sigma_2 = \Sigma'$ combine to imply that $K_{F,\Sigma}^{\Sigma',\bullet}$ is acyclic outside degrees zero and one and has cohomology as described in claim (iii).

To prove claim (iv) we use the fact that, since each place in Σ' is unramified in F/k, there is an exact sequence of G-modules of the form

$$0 \to \bigoplus_{v\in\Sigma'} P_v \xrightarrow{(d_v)_v} \bigoplus_{v\in\Sigma'} P_v \xrightarrow{\pi} \mathbb{F}_{\Sigma'}^\times \to 0.$$

Here each P_v is $\mathbb{Z}[G]$, the differential d_v maps each element x of P_v to $(1 - \mathrm{N}v \cdot \sigma_v^{-1})(x)$ and the map π is obtained by choosing for each $v \in \Sigma'$ a generator y_v of the cyclic group $\kappa(w_v)^\times$ and then mapping the identity element of P_v to y_v.

This resolution implies $K_{\Sigma'}^\bullet$ belongs to $D^{\mathrm{perf}}(\mathbb{Z}[G])$ and that in $K_0(\mathbb{Z}[G], \mathbb{Q}[G])$ one has

$$\chi_G(K_{\Sigma'}^\bullet, 0) = \delta_G(\prod\nolimits_{v\in\Sigma'} \mathrm{Nrd}_{\mathbb{Q}[G]}([(1 - \mathrm{N}v \cdot \sigma_v)^\#]_r^{-1})) = \delta_G\big(Z^*(f_\Sigma, 1)^\# / Z_{\Sigma'}^*(f_\Sigma, 1)^\#\big).$$

Given these observations, the general property (17) applies to the triangle (48) to imply

$$\begin{aligned} \chi_G(K_{F,\Sigma}^{\Sigma',\bullet}, \epsilon_{F,\Sigma}) - \chi_G(R\Gamma(C_{F,\mathrm{W\acute{e}t}}^\Sigma, \mathbb{G}_m), \epsilon_{F,\Sigma}) &= -\chi_G(K_{\Sigma'}^\bullet, 0) \\ &= \delta_G\big(Z_{\Sigma'}^*(f_\Sigma, 1)^\#\big) - \delta_G\big(Z^*(f_\Sigma, 1)^\#\big). \end{aligned}$$

This shows that the first equality in claim (iv) follows directly from that of Theorem 7.1. This equality then implies that

$$\begin{aligned} \delta_G\big(Z_{\Sigma'}^*(f_\Sigma, 1)\big) &= \delta_G\big((Z_{\Sigma'}^*(f_\Sigma, 1)^\#)^\#\big) \\ &= -\psi_*\big(\delta_G\big(Z_{\Sigma'}^*(f_\Sigma, 1)^\#\big)\big) \\ &= -\psi_*\big(\chi_G(K_{F,\Sigma}^{\Sigma',\bullet}, \epsilon_{F,\Sigma})\big) \\ &= -\chi_G\big((K_{F,\Sigma}^{\Sigma',\bullet})^*, \epsilon_{F,\Sigma}^*\big) \\ &= \chi_G\big((K_{F,\Sigma}^{\Sigma',\bullet})^*[-1], \epsilon_{F,\Sigma}^*\big), \end{aligned}$$

where the second equality follows from (41). This verifies the second equality in claim (iv) and so means that the proof will be complete if we can show that the first square in the diagram (49) commutes, as claimed above.

For any finitely generated G-module M we write D_M for the subcategory of $D(\mathbb{Z}[G])$ comprising bounded complexes C that are acyclic in all degree less than one and such that $H^1(C)$ identifies with the profinite completion $M^\wedge$ of M. Then, since the quotient $M^\wedge/M$ is an injective $\mathbb{Z}[G]$-module there exists a unique morphism θ in $\mathrm{Hom}_{D(\mathbb{Z}[G])}(C,(M^\wedge/M)[-1])$ such that $H^1(\theta)$ is the natural projection $M^\wedge \to M^\wedge/M$. We write $C^\flat$ for the mapping fibre of this morphism (regarding the module M as being clear from context) and note that $H^1(C^\flat) = M$ and that $H^i(C^\flat) = H^i(C)$ for all $i \neq 1$.

For any finite set of places Σ_1 of k that contains Σ we define complexes

$$X_{\Sigma_1} := R\Gamma((C_F^{\Sigma_1})_{\text{ét}}, \mathbb{G}_m) \text{ and } Y_{\Sigma_1} := R\Gamma_c((C_F^{\Sigma_1})_{\text{ét}}, \mathbb{Z}).$$

Then, from the computations in [5, §3.2], one knows that $X_{\Sigma_1}^\vee[-3]$ and $Y_{\Sigma_1}^{\vee\vee}$ both belong to $D_{W_{\Sigma_1}}$ with W_{Σ_1} the $\mathbb{Z}$-linear dual of the cokernel of the diagonal map $\mathbb{Z} \to B^*_{F,\Sigma_1}$, and that the Artin-Verdier Duality Theorem induces a canonical isomorphism $\Delta^\flat_{\Sigma_1} : (Y_{\Sigma_1}^{\vee\vee})^\flat \cong (X_{\Sigma_1}^\vee[-3])^\flat$ in $D(\mathbb{Z}[G])$.

We now fix a finite set Σ'' of places of k that is disjoint from $\Sigma\cup\Sigma'$ and such that $\mathrm{Cl}_{\Sigma_1}(F)$ vanishes with $\Sigma_1 := \Sigma\cup\Sigma''$ and consider the diagram

$$\begin{array}{ccccccc} \big(\mathbb{F}_{\Sigma'}^\times\big)^\vee[-3] & \xrightarrow{\theta_1} & (Y_{\Sigma_1}^{\vee\vee})^\flat & \xrightarrow{\theta_2} & (Y_{\Sigma}^{\vee\vee})^\flat & \xrightarrow{\theta_3} & R\Gamma_c((\mathcal{O}_{F,\Sigma})_{\mathcal{W}},\mathbb{Z}) \\ {\scriptstyle \iota[-2]}\downarrow & & {\scriptstyle \Delta^\flat_{\Sigma_1}}\downarrow & & {\scriptstyle \Delta^\flat_{\Sigma}}\downarrow & & {\scriptstyle (\Delta_\Sigma)^*[-2]}\downarrow \\ (K^\bullet_{\Sigma'})^*[-2] & \xrightarrow{\theta_4} & (X_{\Sigma_1}^\vee[-3])^\flat & \xrightarrow{\theta_5} & (X_{\Sigma}^\vee[-3])^\flat & \xrightarrow{\theta_6} & R\Gamma(C^\Sigma_{F,\mathrm{W\acute{e}t}},\mathbb{G}_m)^*[-2], \end{array} \tag{50}$$

in which we have used the morphisms θ_i specified below.

The choice of Σ_1 ensures (via the argument of [8, p. 573]) that the homomorphism

$$\mathrm{Hom}_{D(\mathbb{Z}[G])}(\big(\mathbb{F}_{\Sigma'}^\times\big)^\vee[-3],(Y_{\Sigma_1}^{\vee\vee})^\flat) \to \mathrm{Hom}_G(\big(\mathbb{F}_{\Sigma'}^\times\big)^\vee, H^3(Y_{\Sigma_1}^{\vee\vee})^\flat) = \mathrm{Hom}_G(\big(\mathbb{F}_{\Sigma'}^\times\big)^\vee,(\mathcal{O}_{F,\Sigma_1}^\times)^\vee)$$

sending each morphism κ to $H^3(\kappa)$ is bijective and so we can define θ_1 to be the morphism for which $H^3(\theta_1)$ is the Pontryagin dual of the natural map $\mathcal{O}_{F,\Sigma_1}^\times \to \mathbb{F}_{\Sigma'}^\times$. Taking θ_4 to be the morphism induced by $(\varrho_{\Sigma_1}^{\Sigma'})^\vee$ the commutativity of the first square in (50) is then clear.

Next we take θ_2 and θ_5 to be induced by the natural morphisms $R\Gamma_c((C_F^{\Sigma_1})_{\text{ét}},\mathbb{Z}) \to R\Gamma_c((C_F^{\Sigma})_{\text{ét}},\mathbb{Z})$ and $R\Gamma((C_F^{\Sigma})_{\text{ét}},\mathbb{G}_m) \to R\Gamma((C_F^{\Sigma_1})_{\text{ét}},\mathbb{G}_m)$ so that the commutativity of the second square in (50) follows from the functoriality of the Artin-Verdier Duality Theorem.

We take θ_3 to be the unique morphism induced by the morphism $Y_\Sigma \to R\Gamma_c((\mathcal{O}_{F,\Sigma})_{\mathcal{W}},\mathbb{Z})$ that occurs in [8, (6)]. In addition, since the complex $((\mathbb{Q}\otimes_\mathbb{Z} W_\Sigma)[-3])^*$ is acyclic, the natural morphism $R\Gamma(C^\Sigma_{F,\mathrm{W\acute{e}t}},\mathbb{G}_m)^*[-2] \to X_\Sigma^*[-2]$ is an isomorphism and we define θ_6 to be the composite of this isomorphism and the morphism $(X_\Sigma^\vee[-3])^\flat \to X^*_\Sigma[-2]$ induced by the natural morphism $X_\Sigma^\vee[-1] \to X_\Sigma^*$. Then, with these definitions, the commutativity of the final square in (50) follows from the construction of the isomorphism Δ_Σ.

Given the commutativity of (50), the required commutativity of the first square in (49) follows from the fact that the very definition of the morphism $\theta''_{\Sigma,\Sigma'}$ in [8] implies it is equal to the composite $\theta_3 \circ \theta_2 \circ \theta_1$ whilst it is straightforward to check $(\varrho_{\Sigma}^{\Sigma'})^*$ is equal to the composite $\theta_6 \circ \theta_5 \circ \theta_4$. □

7.3.3. We end this section by noting that Proposition 7.4 has an interesting consequence concerning the Galois structure of Selmer groups.

Let R be a Dedekind domain. Then by a 'presentation' h of a finitely generated $R[G]$-module M we mean an exact sequence of $R[G]$-modules of the form

$$R[G]^{m_1(h)} \xrightarrow{h} R[G]^{m_2(h)} \to M \to 0$$

where the natural numbers $m_1(h)$ and $m_2(h)$ satisfy $m_1(h) \geq m_2(h)$.

We write $R_\wp$ for the completion of R at a maximal ideal $\wp$. Then an 'adelic presentation' h of an $R[G]$-module M is a collection over all maximal ideals $\wp$ of R of a presentation $h_\wp$ of the $R_\wp[G]$-module $R_\wp \otimes_R M$ with the property that the integers $m_1(h) := m_1(h_\wp)$ and $m_2(h) := m_2(h_\wp)$ are independent of $\wp$ and there exists a homomorphism of $R[G]$-modules $h_R : R[G]^{m_1(h)} \to R[G]^{m_2(h)}$ with the property that $h_\wp = R_\wp \otimes_R h_R$ for almost all $\wp$.

We say that an $R[G]$-module is 'quadratic', respectively 'locally quadratic', if there exists a presentation, respectively adelic presentation, h of M such that $m_1(h) = m_2(h)$. This condition means that, with respect to the given presentation, the $R[G]$-module M, respectively each $R_\wp[G]$-module $M_\wp$, has the same number of generators and relations.

Corollary 7.5. *Fix finite sets of places Σ and Σ' of k as in Proposition 7.4. Then the following claims are valid for the G-module $\mathrm{Sel}_{\Sigma}^{\Sigma'}(F)$.*

(i) *$\mathrm{Sel}_{\Sigma}^{\Sigma'}(F)$ is locally quadratic.*
(ii) *If $L^{\mathrm{Artin},*}(C_F^{\Sigma}, \chi, 1)$ is positive for every χ in $\mathrm{Ir}^{\mathrm{s}}(G)$, then $\mathrm{Sel}_{\Sigma}^{\Sigma'}(F)$ is quadratic.*

Proof. The results of Proposition 7.4(i) and (ii) combine to imply that the complex $K^\bullet := R\,\mathrm{Hom}_{\mathbb{Z}}(R\Gamma_{\Sigma'}(C^{\Sigma}_{F,\mathrm{W\acute{e}t}}, \mathbb{G}_m), \mathbb{Z})$ belongs to $D^{\mathrm{perf}}(\mathbb{Z}[G])$, is acyclic outside degrees one and two and such that $H^1(K^\bullet)$ is torsion-free, $H^2(K^\bullet) = \mathrm{Sel}_{\Sigma}^{\Sigma'}(F)$ and the $\mathbb{Q}[G]$-modules that are respectively spanned by $H^1(K^\bullet)$ and $H^2(K^\bullet)$ are isomorphic.

By a standard argument (as used, for example, in the proof of [2, §5.3, Lem. 6]), these facts combine to imply $K^\bullet$ is isomorphic in $D(\mathbb{Z}[G])$ to a complex of the form $P \xrightarrow{d} \mathbb{Z}[G]^m$, where m is any large enough natural number and P is a projective module such that the $\mathbb{Z}_p[G]$-module $\mathbb{Z}_p \otimes_{\mathbb{Z}} P$ is free of rank m for every prime p.

By Roiter's Lemma [17, (31.6)], the latter fact implies P contains a submodule P' of finite index that is isomorphic to $\mathbb{Z}[G]^m$. We can therefore obtain a quadratic adelic presentation h of $\mathrm{Sel}_{\Sigma}^{\Sigma'}(F)$ by setting $h_p := \mathbb{Z}_p \otimes_{\mathbb{Z}} d$ for all p and taking $h_{\mathbb{Z}}$ to be the restriction of h to P'. This proves claim (i).

Turning to claim (ii) we note that, since $\mathrm{Nrd}_{\mathbb{Q}[G]}([1 - \mathrm{N}v \cdot \sigma_v]_r)$ belongs to $\mathrm{im}(\mathrm{Nrd}_{\mathbb{Q}[G]})$ for every v in Σ', the given hypothesis implies that the leading term $Z^*_{\Sigma'}(f_\Sigma, 1)$ belongs to $\mathrm{im}(\mathrm{Nrd}_{\mathbb{Q}[G]})$. From the second displayed equality in Proposition 7.4(iv), we can therefore deduce that the Euler characteristic of $K^\bullet$ in $K_0(\mathbb{Z}[G])$ vanishes.

Given this fact, the argument used at the end of §6.2.2 implies that the projective module P that occurs in the above construction must be free of rank m. By taking $P' = P$, we therefore obtain a quadratic presentation h of the G-module $\mathrm{Sel}_{\Sigma}^{\Sigma'}(F)$, as required to prove claim (ii). □

7.4. **Refined Stark Conjectures.** In this section we use Theorem 7.1, as interpreted in Proposition 7.4(iv), to prove several previously formulated refinements of Stark's Conjecture. In particular, in this way we shall obtain proofs of Corollaries 1.6 and 1.7.

7.4.1. For the reader's convenience, we first quickly review the theory of non-commutative Fitting invariants introduced by Nickel (cf. [36, §1.0.3]).

Let R be a Dedekind domain with fraction field Q of characteristic zero (so that the algebra $Q[G]$ is semisimple). Then the R-submodule $\xi(R[G])$ of $\zeta(Q[G])$ that is generated by the elements $\mathrm{Nrd}_{Q[G]}(N)$ as N runs over all matrices in $\bigcup_{n\ge 0}\mathrm{M}_n(R[G])$ is an R-order in $\zeta(Q[G])$.

If h is a presentation (in the sense of §7.3.3) of an $R[G]$-module, then its Fitting invariant $\mathrm{Fit}^0_{R[G]}(h)$ is defined to be the $\xi(R[G])$-submodule of $\zeta(Q[G])$ that is generated by the reduced norms (over $Q[G]$) of the $m_2(h)\times m_2(h)$-minors of the matrix of h with respect to the standard bases of $R[G]^{m_1(h)}$ and $R[G]^{m_2(h)}$.

If h is an adelic presentation, then its Fitting invariant is the (finitely generated) $\xi(R[G])$-submodule of $\zeta(Q[G])$ that is obtained by setting

$$\mathrm{Fit}^0_{R[G]}(h) := \bigcap_{\wp}\big(\zeta(Q[G])\cap \mathrm{Fit}^0_{R_\wp[G]}(h_\wp)\big),$$

where $\wp$ runs over all maximal ideals of R.

It is easily checked that if G is abelian, and h is an adelic presentation of an $R[G]$-module M, then $\mathrm{Fit}^0_{R[G]}(h)$ coincides with the initial Fitting ideal $\mathrm{Fit}^0_{R[G]}(M)$ of M, as discussed by Northcott in [37].

Remark 7.6. In [36, §1.0.3] Nickel also defines a canonical ideal $\mathcal{H}(R[G])$ of $\zeta(R[G])$ that has the following property: if h is an adelic presentation of an $R[G]$-module M, then for any element x of $\mathcal{H}(R[G])$ and any element y of $\mathrm{Fit}^0_{R[G]}(h)$ the product $x\cdot y$ is contained in $R[G]$ and annihilates M (by [36, Th. 1.2]). This ideal $\mathcal{H}(R[G])$ is easily seen to be invariant under the R-linear anti-involution of $R[G]$ that inverts elements of G and has also been described explicitly in many cases by Johnston and Nickel in [29]. For example, the argument of [29, Prop. 4.8] implies that if $\wp$ is any maximal ideal of R with residue characteristic prime to the order of the commutator subgroup of G, then one has $\mathcal{H}(R[G])_\wp = \zeta(R_\wp[G])$.

7.4.2. In this section we show Theorem 7.1 extends the known validity of the Brumer-Stark Conjecture for global function fields. We recall that the latter conjecture was first proved by Deligne using 1-motives (for Tate's exposition of this proof see [46, Chap. V]) and then later by Hayes [28] by using rank one Drinfeld modules.

For a non-empty set of places Σ' of k we define the Σ'-modified degree zero divisor class group $\mathrm{Pic}^{\Sigma',0}(F)$ of F to be the cokernel of the natural divisor map $\mathrm{div}^0_{F,\Sigma'}$ from $F^\times_{\Sigma'}$ to the group of degree zero divisors of F with support outside Σ'.

Theorem 7.7. *There exists an adelic presentation h' of the G-module $\mathrm{Pic}^{\Sigma',0}(F)^\vee$ such that*

$$Z_{\Sigma'}(f_\Sigma,1)\in \mathrm{Fit}^0_{\mathbb{Z}[G]}(h').$$

Proof. We claim first that for the quadratic adelic presentation h of $\mathrm{Sel}_{\Sigma}^{\Sigma'}(F)$ that is constructed in the proof of Corollary 7.5 one has

$$Z_{\Sigma'}(f_\Sigma,1)\in \mathrm{Fit}^0_{\mathbb{Z}[G]}(h). \tag{51}$$

To prove this it is enough to show $Z_{\Sigma'}(f_\Sigma,1)$ belongs to $\mathrm{Fit}^0_{\mathbb{Z}_p[G]}(h_p)$ for every prime p.

But if $\mathbb{Z}_p[G]^m \xrightarrow{d_p} \mathbb{Z}_p[G]^m$ is the representative of the complex $\mathbb{Z}_p\otimes_{\mathbb{Z}}K^\bullet$ that is constructed in the above argument, then $\mathrm{Fit}^0_{\mathbb{Z}_p[G]}(h_p)$ is equal to $\mathrm{Fit}^0_{\mathbb{Z}_p[G]}(d_p)=\xi(\mathbb{Z}_p[G])\cdot \mathrm{Nrd}_{\mathbb{Q}_p[G]}(d_p)$.

In addition, the second equality of Proposition 7.4(iv) combines with the commutativity of the diagram (36) to imply that

$$\begin{aligned}\delta_{\mathbb{Z}_p[G]}(\iota_p(Z^*_{\Sigma'}(f_\Sigma,1))) &= \pi_p(\chi_G(K^\bullet[-1],\epsilon^*_{F,\Sigma}))\\ &= \chi_{\mathbb{Z}_p[G]}(\mathbb{Z}_p\otimes_{\mathbb{Z}}K^\bullet[-1],\mathbb{Q}_p\otimes_{\mathbb{Q}}\epsilon^*_{F,\Sigma})\\ &= \delta_{\mathbb{Z}_p[G]}\big(\mathrm{Nrd}_{\mathbb{Q}_p[G]}(d_p^\dagger)\big).\end{aligned}$$

Here $d_p^\dagger$ is an automorphism of the $\mathbb{Q}_p[G]$-module $\mathbb{Q}_p[G]^m$ that agrees with $\mathbb{Q}_p\otimes_{\mathbb{Z}_p}d_p$ on some choice of $\mathbb{Q}_p[G]$-equivariant direct complement to $\mathbb{Q}_p\cdot\ker(d_p)$ in $\mathbb{Q}_p[G]^m$ and the last equality is verified by an explicit computation of the non-abelian determinant (as per [4, Lem. A.1.1]). From this displayed equality we deduce the existence of an element u of $\ker(\delta_{\mathbb{Z}_p[G]})=\mathrm{Nrd}_{\mathbb{Q}_p[G]}(K_1(\mathbb{Z}_p[G]))\subset\xi(\mathbb{Z}_p[G])$ such that

$$Z^*_{\Sigma'}(f_\Sigma,1)=u\cdot\mathrm{Nrd}_{\mathbb{Q}_p[G]}(d_p^\dagger).$$

Next we recall (from, for example, [46, Chap. I, Prop. 3.4]) that for each χ in $\mathrm{Ir}(G)$ the order of vanishing $r_{f_\Sigma,\chi}$ at $t=1$ of the series $L^{\mathrm{Artin}}(C_F^\Sigma,\chi,t)$ is equal to

$$r_{f_\Sigma,\chi}=\dim_{\mathbb{Q}^c}\big(\mathrm{Hom}_{\mathbb{Q}^c[G]}(V_{\check\chi},\mathbb{Q}^c\otimes_{\mathbb{Z}}B^0_{F,\Sigma})\big)=\dim_{\mathbb{Q}^c}\big(\mathrm{Hom}_{\mathbb{Q}^c[G]}(V_{\check\chi},\mathbb{Q}^c\otimes_{\mathbb{Z}}\mathcal{O}^\times_{F,\Sigma})\big), \tag{52}$$

where $V_{\check\chi}$ is a choice of $\mathbb{Q}^c[G]$-module with character the contragredient $\check\chi$ of χ and the second equality is a consequence of the isomorphism $\mathbb{Q}\otimes_{\mathbb{Z}}D^q_{F,\Sigma}$.

We write e for the sum of all primitive idempotents of $\zeta(\mathbb{Q}[G])$ that annihilate the space $\mathbb{Q}_p\otimes_{\mathbb{Z}_p}\ker(d_p)\cong\mathrm{Hom}_{\mathbb{Z}}(B^{0,*}_{F,\Sigma},\mathbb{Q}_p)$. Then, since

$$\dim_{\mathbb{Q}^c}\big(\mathrm{Hom}_{\mathbb{Q}^c[G]}(V_{\check\chi},\mathbb{Q}^c\otimes_{\mathbb{Z}}B^0_{F,\Sigma})\big)=\dim_{\mathbb{Q}^c}\big(\mathrm{Hom}_{\mathbb{Q}^c[G]}(V_\chi,\mathrm{Hom}_{\mathbb{Z}}(B^0_{F,\Sigma},\mathbb{Q}^c))\big),$$

the formula (52) implies both that $Z_{\Sigma'}(f_\Sigma,1)=e\cdot Z^*_{\Sigma'}(f_\Sigma,1)$ and also that $\mathrm{Nrd}_{\mathbb{Q}_p[G]}(d_p)=e\cdot\mathrm{Nrd}_{\mathbb{Q}_p[G]}(d_p^\dagger)$.

It follows that $Z_{\Sigma'}(f_\Sigma,1)$ is equal to

$$e\cdot Z^*_{\Sigma'}(f_\Sigma,1)=eu\cdot\mathrm{Nrd}_{\mathbb{Q}_p[G]}(d_p^\dagger)=u\cdot\mathrm{Nrd}_{\mathbb{Q}_p[G]}(d_p)$$

and so belongs to $\mathrm{Fit}^0_{\mathbb{Z}_p[G]}(h_p)$, as suffices to prove (51).

Given this containment, the argument of [36, Prop. 3.5(i)] shows that the claimed result will follow if $\mathrm{Pic}^{\Sigma',0}(F)^\vee$ is isomorphic, as a G-module, to a quotient of $\mathrm{Sel}_{\Sigma}^{\Sigma'}(F)$.

To verify this we write $\theta_{\Sigma'}$ for the homomorphism defined as in (46) with $\Sigma_2=\Sigma'$ and Σ_1 empty. Then there is a natural surjection from $\mathrm{Sel}_{\Sigma}^{\Sigma'}(F)$ to $\mathrm{cok}(\theta_{\Sigma'})$ and so it suffices to prove that the latter group identifies with $\mathrm{Pic}^{\Sigma',0}(F)^\vee$.

To do this we note that $\mathrm{div}_{F,\Sigma'}$ is injective (as $F^\times_{\Sigma'}$ is torsion-free) and that $\mathrm{Pic}^{\Sigma',0}(F)$ is the (finite) torsion subgroup of $\mathrm{cok}(\mathrm{div}_{F,\Sigma'})$. Hence, if we write $\mathrm{Div}_{\Sigma'}(F)$ for the group of divisors of F with support outside Σ'_F, then there exists a natural short exact sequence $0 \to F^\times_{\Sigma'} \to \mathrm{Div}_{\Sigma'}(F) \to \mathrm{cok}(\mathrm{div}_{F,\Sigma'}) \to 0$. Since, by applying the functor $\mathrm{Hom}_{\mathbb{Z}}(-,\mathbb{Z})$ to this sequence, one obtains an exact sequence

$$\prod_{w\notin\Sigma'_F}\mathbb{Z} \xrightarrow{\theta_{\Sigma'}} \mathrm{Hom}_{\mathbb{Z}}(F^\times_{\Sigma'},\mathbb{Z}) \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{cok}(\mathrm{div}_{F,\Sigma'}),\mathbb{Z}) \to 0$$

it is therefore enough to note that $\mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{cok}(\mathrm{div}_{F,\Sigma'}),\mathbb{Z})$ identifies with

$$\mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{cok}(\mathrm{div}_{F,\Sigma'})_{\mathrm{tor}},\mathbb{Z}) = \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{Pic}^{\Sigma',0}(F),\mathbb{Z}) \cong \mathrm{Pic}^{\Sigma',0}(F)^\vee.$$

□

Remark 7.8. After taking account of Remark 7.6, it is clear that the containment of Theorem 7.7 implies a refined version of the 'non-abelian Brumer Stark Conjecture' formulated by Nickel in [36, Conj. 2.6]. In addition, if G is abelian, then for any presentation h' as in Theorem 7.7, the ideal $\mathrm{Fit}^0_{\mathbb{Z}[G]}(h')$ is equal to $\mathrm{Fit}^0_{\mathbb{Z}[G]}(\mathrm{Pic}^{\Sigma',0}(F)^\vee)$ and so annihilates $\mathrm{Pic}^{\Sigma',0}(F)^\vee$. It follows that Theorem 7.7 also specialises to recover the original Brumer-Stark Conjecture for global function fields, as proved by Deligne in [46, Chap. V] and by Hayes in [28].

7.4.3. We now write Σ_{sp} for the subset of Σ comprising places that split completely in F and set $a = a_{F/k,\Sigma} := |\Sigma_{\mathrm{sp}}|$.

If Σ_{sp} is a non-empty proper subset of Σ, then the formula (52) implies that $Z_{\Sigma'}(f_\Sigma,1)$ vanishes and so the statement of Theorem 7.7 is trivial.

However, in any such case, the formula (52) also implies that the 'a-th equivariant derivative' series

$$Z^{(a)}_{\Sigma'}(f_\Sigma,t) := \Big(\sum\nolimits_{\chi\in\mathrm{Ir}(G)}(1-t)^{-a\chi(1)}\cdot e_\chi\Big)\cdot Z_{\Sigma'}(f_\Sigma,t)$$

is holomorphic at $t=1$ and that its value $Z^{(a)}_{\Sigma'}(f_\Sigma,1)$ at $t=1$ does not, in general, vanish.

In the next result we prove that this value encodes information about the Galois structure of ideal class groups, thereby proving a conjecture of the first author from [4].

For any homomorphism of G-modules $\phi : \mathcal{O}^\times_{F,\Sigma,\Sigma'} \to B^0_{F,\Sigma}$ we set

$$R(\phi) := \mathrm{Nrd}_{\mathbb{Q}[G]}((\mathbb{Q}\otimes_{\mathbb{Z}} D^q_{F,\Sigma})^{-1}\circ(\mathbb{Q}\otimes_{\mathbb{Z}}\phi)) \in \zeta(\mathbb{Q}[G])$$

where the isomorphism $\mathbb{Q}\otimes_{\mathbb{Z}} D^q_{F,\Sigma}$ is as defined in (39).

In this result we also use the ideal $\mathcal{H}(\mathbb{Z}[G])$ of $\zeta(\mathbb{Z}[G])$ that was discussed in Remark 7.6.

Theorem 7.9. *If Σ_{sp} is a proper subset of Σ, then the following claims are valid for every ϕ in $\mathrm{Hom}_G(\mathcal{O}^\times_{F,\Sigma,\Sigma'}, B^0_{F,\Sigma})$.*

(i) *If Σ_1 is any subset of Σ that properly contains Σ_{sp}, then for any element x of $\mathcal{H}(\mathbb{Z}[G])$ the product $x\cdot Z^{(a)}_{\Sigma'}(f_\Sigma,1)\cdot R(\phi)$ belongs to $\mathbb{Z}[G]$ and annihilates $\mathrm{Cl}^{\Sigma'}_{\Sigma_1}(F)$.*

(ii) *If G is abelian, then $Z^{(a)}_{\Sigma'}(f_\Sigma,1)\cdot R(\phi)$ belongs to $\mathrm{Fit}^a_{\mathbb{Z}[G]}(\mathrm{Pic}^{\Sigma',0}(F)^\vee)^{\#}$.*

Proof. To prove the stated claims it is enough to show for every prime p that the product $x\cdot Z^{(a)}_{\Sigma'}(f_\Sigma,1)\cdot R(\phi)$ belongs to $\mathbb{Z}_p[G]$ and annihilates $\mathbb{Z}_p\otimes_\mathbb{Z}\mathrm{Cl}^{\Sigma'}_{\Sigma_1}(F)$ and that, if G is abelian, then $Z^{(a)}_{\Sigma'}(f_\Sigma,1)\cdot R(\phi)$ also belongs to $\mathrm{Fit}^a_{\mathbb{Z}_p[G]}(\mathbb{Z}_p\otimes_\mathbb{Z}\mathrm{Pic}^{\Sigma',0}(F)^\vee)^\#$.

To prove both of these claims we fix p and set $K^\bullet := \mathbb{Z}_p\otimes_\mathbb{Z} R\Gamma_{\Sigma'}(C^\Sigma_{F,\mathrm{W\acute{e}t}},\mathbb{G}_m)$, $U := \mathbb{Z}_p\otimes_\mathbb{Z}\mathcal{O}^\times_{F,\Sigma,\Sigma'}$, $B := \mathbb{Z}_p\otimes_\mathbb{Z} B^0_{F,\Sigma}$, $B_1 := \mathbb{Z}_p\otimes_\mathbb{Z} B_{F,\Sigma_{\mathrm{sp}}}$ and $N := \mathbb{Z}_p\otimes_\mathbb{Z}\mathrm{Sel}^{\Sigma'}_\Sigma(F)$. We also write M^* for the $\mathbb{Z}_p$-linear dual of a $\mathbb{Z}_p[G]$-module M, regarded as endowed with contragredient action of $\mathbb{Z}_p[G]$.

Then, since U is torsion-free, the approach used in the proof of Corollary 7.5 combines with Proposition 7.4(i) and (iii) to imply that $K^\bullet$ is represented by a complex of $\mathbb{Z}_p[G]$-modules of the form $P\xrightarrow{\vartheta}P$ where P is a free of rank m (for any large enough m) and the first term is placed in degree zero.

We label the places in Σ_{sp} as $\{v_i\}_{1\le i\le a}$ and for each index i choose a place w_i of F above v_i. We note that B_1 is then a free G-module of rank a with basis $\{w_i\}_{1\le i\le a}$. Since Σ_{sp} is assumed to be a proper subset of Σ, the natural projection map $B^0\to B_1$ is surjective. We may therefore fix a basis $\{b_i\}_{1\le i\le m}$ of P so that the natural projection map

$$P\to\mathrm{cok}(\vartheta)\cong H^1(K^\bullet)\twoheadrightarrow H^1(K^\bullet)_{\mathrm{tf}}\cong B\twoheadrightarrow B_1$$

sends b_i to w_i if $1\le i\le a$ and to zero otherwise.

In addition, since $\mathrm{im}(\vartheta)$ is torsion-free and the G-module B_1 is free, the restriction map

$$\mathrm{Hom}_{\mathbb{Z}_p[G]}(P,B_1)\to\mathrm{Hom}_{\mathbb{Z}_p[G]}(\ker(\vartheta),B_1)=\mathrm{Hom}_{\mathbb{Z}_p[G]}(U,B_1)$$

is surjective. We can therefore choose an element $\tilde\phi$ of $\mathrm{Hom}_{\mathbb{Z}_p[G]}(P,B_1)$ that maps to the composite of $\mathbb{Z}_p\otimes_\mathbb{Z}\phi$ and the projection $B^0\to B_1$ and consider the composite homomorphism of $\mathbb{Z}_p[G]$-modules

$$\phi' : P\to B_1\to P,$$

where the first map is $\tilde\phi$ and the second sends each place w_i to b_i (for $1\le i\le a$).

We write $M(\phi'+\vartheta)$ for the matrix in $\mathrm{M}_m(\mathbb{Z}_p[G])$ that represents the endomorphism $\phi'+\vartheta$ of P with respect to the given basis $\{b_i\}_{1\le i\le m}$. Then the argument of [4, Prop. 7.2.1(ii) and Lem. 7.3.1] implies the existence of an element u of $\mathrm{Nrd}_{\mathbb{Q}_p[G]}(K_1(\mathbb{Z}_p[G]))\subset\xi(\mathbb{Z}_p[G])$ such that

$$Z^{(a)}_{\Sigma'}(f_\Sigma,1)\cdot R(\phi)=u\cdot\mathrm{Nrd}_{\mathbb{Q}_p[G]}(M(\phi'+\vartheta)). \tag{53}$$

To interpret this equality we write b_i^* for each index i for the element of P^* that is dual to b_i. For each homomorphism ψ in $\mathrm{Hom}_{\mathbb{Z}_p[G]}(U,B^0)$ and each index i with $1\le i\le a$ we then write ψ_i for the element of U^* that sends each element x to the coefficient of w_i that occurs in $\psi(x)$.

We use Proposition 7.4(ii) to identify the cohomology groups of the complex $K_1^\bullet := R\,\mathrm{Hom}_{\mathbb{Z}_p}(K^\bullet,\mathbb{Z}_p)$ and then consider the following exact triangle in $D^{\mathrm{p}}(\mathbb{Z}_p[G])$

$$\mathbb{Z}_p[G]^{\oplus a,\bullet}\xrightarrow{\theta}K_1^\bullet[-1]\to K_2^\bullet\to\mathbb{Z}_p[G]^{\oplus a,\bullet}[1]. \tag{54}$$

Here $\mathbb{Z}_p[G]^{\oplus a,\bullet}$ denotes the complex $\mathbb{Z}_p[G]^{\oplus a}[0]\oplus\mathbb{Z}_p[G]^{\oplus a}[-1]$ and, writing $\{c_i\}_{1\le i\le a}$ for the canonical basis of $\mathbb{Z}_p[G]^{\oplus a}$, the morphism θ in $D(\mathbb{Z}_p[G])$ is uniquely specified by the condition that for each index i one has

$$H^j(\theta)(c_i) = \begin{cases} w_i^* \in B_1^* \subset (B^0)^* = H^0(K_1^\bullet[-1]), & \text{if } j = 0 \\ \widetilde{\phi}_i \in N = H^1(K_1^\bullet[-1]), & \text{if } j = 1, \end{cases}$$

where w_i^* is the element of B_1^* that is dual to w_i and $\tilde{\phi}_i$ is a choice of pre-image of ϕ_i under the projection map $N \to U^*$ that is induced by (47).

Following Lemma 7.10 below, we may, and will, henceforth assume that the G-module spanned by the homomorphisms $\{\phi_i\}_{1\le i\le a}$ is free of rank a. Then, in this case, the long exact cohomology sequence of the exact triangle (54) implies $K_2^\bullet$ is acyclic outside degrees zero and one and induces identifications $H^0(K_2^\bullet) = (\mathbb{Z}_p \otimes_{\mathbb{Z}} B^0_{F,\Sigma\setminus\Sigma_{\rm sp}})^*$ and $H^1(K_2^\bullet) = N/\mathcal{E}$, where $\mathcal{E}$ denotes the (free, rank a) G-submodule of N that is generated by the set $\{\tilde{\phi}_i\}_{1\le i\le a}$.

We now consider the following diagram of G-modules

$$\begin{array}{ccccc} P_1^* & \xrightarrow{\ \rm id\ } & P_1^* & & \\ \uparrow{\scriptstyle \rm id} & & \uparrow{\scriptstyle (-{\rm id},\pi)} & & \\ P_1^* & \xrightarrow{(0,\iota)} & P_1^* \oplus P^* & \xrightarrow{((\phi')^*\circ\iota,\vartheta^*)} & P^* \\ & & {\scriptstyle (\pi,{\rm id})}\uparrow & & \uparrow{\scriptstyle \rm id} \\ & & P^* & \xrightarrow{(\phi'+\vartheta)^*} & P^*. \end{array}$$

Here P_1 denotes the free G-submodule of P spanned by $\{b_i\}_{1\le i\le a}$ and ι and π are respectively the natural inclusion $P_1^* \to P^*$ and projection $P^* \to P_1^*$ maps.

Now $K_1^\bullet[-1]$ is represented by the complex $P^* \xrightarrow{\vartheta^*} P^*$, where the first term is placed in degree zero and the identification of $\ker(\vartheta^*)$ with $H^0(K_1^\bullet[-1]) = (B^0)^*$ is such that w_i^* corresponds to the element b_i^* of P^* for each i with $1 \le i \le a$. For this reason, the central row of the above diagram represents the mapping cone of the morphism θ in the exact triangle (54) and so is isomorphic in $D(\mathbb{Z}_p[G])$ to the complex $K_2^\bullet$.

In addition, the above diagram commutes (since the construction of ϕ' implies $(\phi')^*$ factors through π) and all of its columns are exact.

Thus, since its upper row is an acyclic complex, the diagram implies that $K_2^\bullet$ is also isomorphic in $D(\mathbb{Z}_p[G])$ to the complex given by the lower row of the diagram. Since $H^1(K_2^\bullet) = N/\mathcal{E}$, this fact gives rise to a presentation of $\mathbb{Z}_p[G]$-modules

$$P^* \xrightarrow{(\phi'+\vartheta)^*} P^* \to N/\mathcal{E} \to 0$$

that we denote by h.

For a matrix M in $\mathrm{M}_m(\mathbb{Z}_p[G])$ we now write $M^\#$ for the matrix in $\mathrm{M}_m(\mathbb{Z}_p[G])$ that is obtained by applying to each entry of M the $\mathbb{Z}_p$-linear automorphism of $\mathbb{Z}_p[G]$ that inverts elements of G.

Then the matrix of the endomorphism $(\phi' + \vartheta)^*$ with respect to the basis $\{b_i^*\}_{1\le i\le m}$ of P^* is the transpose of $M(\phi' + \vartheta)^\#$ and so the equality (53) implies that

$$Z^{(a)}_{\Sigma'}(f_\Sigma, 1) \cdot R(\phi) \in \mathrm{Fit}^0_{\mathbb{Z}_p[G]}(h)^\#. \tag{55}$$

We now write ϱ for the natural surjective homomorphism from $N = \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathrm{Sel}_{\Sigma}^{\Sigma'}(F)$ to $N_1 := \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathrm{Sel}_{\Sigma_1}^{\Sigma'}(F)$. Then [36, Prop. 3.5(i)] implies the existence of a presentation h_1 of the $\mathbb{Z}_p[G]$-module $N_1/\varrho(\mathcal{E})$ such that $\mathrm{Fit}^0_{\mathbb{Z}_p[G]}(h) \subseteq \mathrm{Fit}^0_{\mathbb{Z}_p[G]}(h_1)$.

The above containment therefore combines with Remark 7.6 to imply that for any element x of $\mathcal{H}(\mathbb{Z}[G])$ the product $x\cdot(Z^{(a)}_{\Sigma'}(f_\Sigma,1)\cdot R(\phi))^{\#}$ belongs to $\mathbb{Z}_p[G]$ and annihilates $N_1/\varrho(\mathcal{E})$.

In addition, since our choice of homomorphisms ϕ_i implies that the G-module $\varrho(\mathcal{E})$ is free (of rank a), the exact sequence (47) with Σ_2 replaced by Σ' implies $\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathrm{Cl}_{\Sigma_1}^{\Sigma'}(F)^\vee$ is isomorphic to a submodule of $N_1/\varrho(\mathcal{E})$.

It follows that $x\cdot(Z^{(a)}_{\Sigma'}(f_\Sigma,1)\cdot R(\phi))^{\#}$ annihilates $\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathrm{Cl}_{\Sigma_1}^{\Sigma'}(F)^\vee$, or equivalently that $x^{\#}\cdot Z^{(a)}_{\Sigma'}(f_\Sigma,1)\cdot R(\phi)$ annihilates $\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathrm{Cl}_{\Sigma_1}^{\Sigma'}(F)$. To complete the proof of claim (i) we therefore need only recall that $\mathcal{H}(\mathbb{Z}[G])^{\#} = \mathcal{H}(\mathbb{Z}[G])$ (cf. Remark 7.6).

Turning to claim (ii), we note that, if G is abelian then (55) implies that

$$Z^{(a)}_{\Sigma'}(f_\Sigma,1)\cdot R(\phi) \in \mathrm{Fit}^0_{\mathbb{Z}_p[G]}(N/\mathcal{E})^{\#} \subseteq \mathrm{Fit}^a_{\mathbb{Z}_p[G]}(N)^{\#}.$$

Here the inclusion follows from a general property of higher Fitting ideals (which is a consequence, for example, of the result of [8, Lem. 7.1] with $b = 0$) and the fact that the G-module $\mathcal{E}$ is generated by a set of a elements.

The above containment then directly implies claim (ii) since $\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathrm{Pic}^{\Sigma',0}(F)^\vee$ is isomorphic to a quotient of N (see the proof of Theorem 7.7) and so $\mathrm{Fit}^a_{\mathbb{Z}_p[G]}(N)$ is contained in $\mathrm{Fit}^a_{\mathbb{Z}_p[G]}(\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathrm{Pic}^{\Sigma',0}(F)^\vee)$. □

Lemma 7.10. *It is enough to prove Theorem 7.9(i) in the case that the homomorphisms $\{\phi_i\}_{1\le i\le a}$ constructed above span a free $\mathbb{Z}_p[G]$-module of rank a.*

Proof. In this argument we use the same notation as in the proof of Theorem 7.9.

Then it suffices to prove the existence of a homomorphism ϕ' in $\mathrm{Hom}_{\mathbb{Z}_p[G]}(U,B^0)$ for which the $\mathbb{Z}_p[G]$-module generated by $\{\phi'_i\}_{1\le i\le a}$ is free of rank a and one also has

$$Z^{(a)}_{\Sigma'}(f_\Sigma,1)\cdot R(\phi') \equiv Z^{(a)}_{\Sigma'}(f_\Sigma,1)\cdot R(\phi) \quad \text{modulo } |\mathrm{Cl}_{\Sigma_1}^{\Sigma'}(F)|\cdot\zeta(\mathbb{Z}_p[G]). \tag{56}$$

To do this we note that, since $Z^{(a)}_{\Sigma'}(f_\Sigma,1)$ belongs to $\zeta(\mathbb{Q}_p[G])$, there exists a natural number n such that $n\cdot Z^{(a)}_{\Sigma'}(f_\Sigma,1) \in |\mathrm{Cl}_{\Sigma_1}^{\Sigma'}(F)|\cdot\zeta(\mathbb{Z}_p[G])$. Then an explicit computation of reduced norms shows that there exists a natural number m_n such that for any ψ in $\mathrm{Hom}_{\mathbb{Z}_p[G]}(U,B^0)$ and any integer m' one has

$$R(\phi + m'm_n\cdot\psi) \equiv R(\phi) \ \text{ modulo } n\cdot\zeta(\mathbb{Z}_p[G])$$

and hence that the congruence (56) is satisfied by $\phi' := \phi + m'm_n\cdot\psi$.

It is therefore enough to prove that for a suitable choice of ψ and m' this homomorphism ϕ' is such that the $\mathbb{Z}_p[G]$-module generated by $\{\phi'_i\}_{1\le i\le a}$ is free of rank a.

Now, since the $\mathbb{Q}_p[G]$-modules spanned by U and B^0 are isomorphic and the natural projection map $B^0 \to B_1$ is surjective, there exists a homomorphism ψ in $\mathrm{Hom}_{\mathbb{Z}_p[G]}(U,B^0)$ so that the $\mathbb{Q}_p[G]$-module V spanned by $\{\psi_i\}_{1\le i\le a}$ is free of rank a.

We fix a $\mathbb{Q}_p[G]$-equivariant section σ to the inclusion $V \subseteq \mathbb{Q}_p\otimes_{\mathbb{Z}_p} U^*$ and, for each integer r, consider the endomorphisms α and α_r of the $\mathbb{Q}_p[G]$-module V that send each ψ_i to $\sigma(\phi_i)$

and to $\sigma(\phi_i + r\cdot\psi_i) = \sigma(\psi_i) + r\cdot\psi_i$ respectively. Write M and M_r for the matrices in $\mathrm{M}_a(\mathbb{Q}_p[G])$ that represent α and α_r with respect to the basis $\{\psi_i\}_{1\le i\le a}$.

Then M_r is equal to $M + r\cdot \mathrm{Id}_a$ and so the matrix M_r is invertible whenever r is large enough to ensure that $-r$ is not an eigenvalue of any simple algebra component of the matrix M. In any such case, therefore, the elements $\{\sigma(\phi_i + r\cdot\psi_i)\}_{1\le i\le a}$ span a free $\mathbb{Q}_p[G]$-module of rank a.

By choosing r to be a suitable multiple of m_n, we therefore obtain a homomorphism $\phi + r\cdot\psi$ that has all of the required properties. □

Theorems 7.1 and 7.9(i) now combine to give the following result.

Corollary 7.11. *The central conjecture of* [4] *is valid for all global function fields.*

Proof. The formulation of [4, Conj. 2.4.1] makes two predictions, the first concerning the existence of matrices with certain properties and the second concerning the annihilation of ideal class groups. The proof of [4, Th. 4.1.1] shows that, in the case of global function fields, the first of these predictions is implied by the validity of [2, Conj. C(K/k)] and hence now follows as a consequence of Theorem 7.1 (and Remark 7.2). The second prediction of [4, Conj. 2.4.1] follows directly from the containment that is proved in Theorem 7.9(i). □

Remark 7.12. The predictions of [4, Conj. 2.4.1] are known to incorporate a natural generalization to non-abelian extensions of many existing refinements of Stark's Conjecture for abelian extensions that are due to, inter alia, Rubin [41], Gross [24] and Tate [47] (for more details see [4, Prop. 3.4.1, Th. 7.5.1 and Rem. 7.5.2].) In addition, the result proved in Theorem 7.9(i) is strictly finer than the corresponding prediction that is made in [4, Conj. 2.4.1] since the latter deals only with ideal class groups of the form $\mathrm{Cl}_{\Sigma_1}(F)$ rather than with the potentially larger groups $\mathrm{Cl}_{\Sigma_1}^{\Sigma'}(F)$.

7.4.4. Turning to the proof of Corollary 1.7, we note first that if the finite set Σ is large enough to ensure $\mathrm{Cl}(\mathcal{O}_{F,\Sigma})$ vanishes, then the identifications in (38) imply $R\Gamma(C^{\Sigma}_{F,\mathrm{W\acute{e}t}}, \mathbb{G}_m)$ corresponds to a canonical element $\epsilon_{F/k,\Sigma}$ of the Yoneda Ext-group $\mathrm{Ext}^2_G(B^0_{F,\Sigma}, \mathcal{O}^\times_{F,\Sigma})$.

In addition, the argument of Flach and the first author in [5, Prop. 3.5] leads (via [2, §4.1, Prop. 4.1]) to an explicit description of $\epsilon_{F/k,\Sigma}$ in terms of the fundamental classes of class field theory that were introduced by Tate in [44].

This fact is in turn the key point in the proof of [2, §4.2, Th. 4.1] which asserts that [2, Conj. C(F/k)] implies the validity of Chinburg's $\Omega(3)$-Conjecture for F/k.

Given the latter implication, it is then clear that Corollary 1.7 follows directly from Theorem 7.1 (and Remark 7.2).

Our final observation now concerns Chinburg's $\Omega(1)$-Conjecture.

Corollary 7.13. *The* $\Omega(1)$*-Conjecture is valid for all tamely ramified Galois extensions of global function fields.*

Proof. Chinburg has proved the validity of his $\Omega(2)$-Conjecture for all tamely ramified extensions F/k (this follows upon combining [10, §4.2, Th. 4] with [13, Cor. 4.10]). In addition, the observations made by Chinburg in [10, §4.1, Th. 2 and the remarks that follow it] imply that the $\Omega(1)$-Conjecture is automatically valid for a Galois extension F/k whenever the $\Omega(2)$-Conjecture and $\Omega(3)$-Conjecture are both valid for F/k. Given these facts, the claimed result therefore follows directly from Corollary 1.7. □

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