IWASAWA THEORY OVER $\mathbb{Z}[\![\mathbb{Z}_p]\!]$ AND WEIL-ÉTALE COHOMOLOGY

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Abstract. For each prime number p, we establish key ring-theoretical properties of the completed group-ring $\mathbb{Z}[\![\mathbb{Z}_p]\!]$ and also strengthen the theory of 'pro-discrete' $\mathbb{Z}[\![\mathbb{Z}_p]\!]$-modules developed in [7]. We then apply the resulting theory to obtain a variety of new, and explicit, results concerning divisor class groups and the Weil-étale cohomology of $\mathbb{G}_m$ over $\mathbb{Z}_p$-extensions of global function fields of characteristic p.

1. Introduction

For a commutative ring A, rational prime p and finite group H, the completed group-ring $A[\![\mathbb{Z}_p \times H]\!]$ is equal to the inverse limit of the group-rings $A[(\mathbb{Z}/(p^n)) \times H]$ with respect to the canonical projection maps $A[(\mathbb{Z}/(p^{n+1})) \times H] \to A[(\mathbb{Z}/(p^n)) \times H]$.

Arithmetic modules over the ring $\mathbb{Z}[\![\mathbb{Z}_p]\!]$ arise naturally from the limits of families of modules in $\mathbb{Z}_p$-extensions of global fields. However, since $\mathbb{Z}[\![\mathbb{Z}_p]\!]$ is neither Noetherian nor has a natural compact topology, such modules are usually considered, following Iwasawa, by passing to pro-p completions and then working over the associated (Noetherian and compact) algebra $\mathbb{Z}_p[\![\mathbb{Z}_p]\!]$. Nevertheless, this passage to pro-p completion can lose significant information and previous authors have either stressed the benefits, and intrinsic difficulties, in concrete situations of working over either $\mathbb{Z}[\![\mathbb{Z}_p]\!]$ or $\mathbb{Z}_\ell[\![\mathbb{Z}_p]\!]$ for a prime $\ell \neq p$, or have tried to understand particular aspects of the problem (see, for instance, Washington [33] and Coleman [10] and, for a more recent example, Bandini and Longhi [3]). With these general issues in mind, in previous work with Daoud [7] we introduced a category of 'pro-discrete' $\mathbb{Z}[\![\mathbb{Z}_p]\!]$-modules for which one can prove analogues of both Nakayama's Lemma and Roiter's Lemma and thereby establish criteria for finite-presentability. By using these results we were then able to prove that the ring $\mathbb{Z}[\![\mathbb{Z}_p]\!]$ is not coherent (in the classical sense of Chase [9] and Bourbaki [5, Chap. 1]) but is 2-coherent (in the sense of Costa [12]) and also has weak Krull dimension 2 (in the sense of Tang [27]). In this article, we further clarify the basic ring-theoretical properties of $\mathbb{Z}[\![\mathbb{Z}_p]\!]$ before extending key aspects of the general theory of pro-discrete modules and then applying the results to study Iwasawa-theoretic modules in the setting of global function fields.

In a little more detail, after reviewing the basic theory of pro-discrete modules in §2, we shall start our current investigations in §3 by studying the interplay between ideals of $\mathbb{Z}[\![\mathbb{Z}_p]\!]$ and of $\mathbb{Z}_\ell[\![\mathbb{Z}_p]\!]$ for each prime ℓ. In this way, for example, we prove $\mathbb{Z}[\![\mathbb{Z}_p]\!]$ has infinite Krull dimension and that ideals of the classical Iwasawa algebra $\mathbb{Z}_p[\![\mathbb{Z}_p]\!]$ are uniquely determined by their pull-backs to $\mathbb{Z}[\![\mathbb{Z}_p]\!]$. We then discuss the maximal spectrum of $\mathbb{Z}[\![\mathbb{Z}_p]\!]$ and, in particular, classify all maximal ideals that are pro-discrete and also show, using a classical result of Zsigmondy, that there are infinitely many maximal ideals whose residue field is a rational vector space of uncountable dimension.

In §4 we then establish criteria for finitely-generated pro-discrete modules over rings of the form $\mathbb{Z}[\![\mathbb{Z}_p \times H]\!]$ with H finite and abelian to be projective and to be torsion. We also obtain a method for computing the initial Fitting ideals of finitely-presented pro-discrete torsion modules over such rings, before establishing basic properties of the derived limits of perfect complexes in the theory of pro-discrete modules. In the remainder of the article, we shall then apply the resulting theory to study $\mathbb{Z}[\![\mathbb{Z}_p \times H]\!]$-modules that naturally arise over $\mathbb{Z}_p$-extensions of global function fields

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of characteristic p from the limits of divisor class groups (in §5) and from the limits of Lichtenbaum's Weil-étale cohomology of $\mathbb{G}_m$ (in §6).

To be more precise about these arithmetic applications, in Theorem 5.1 we prove that, if only finitely many places ramify in a given $\mathbb{Z}_p$-extension, then the inverse limit of divisor class groups over this extension is both pro-discrete and torsion over $\mathbb{Z}[\![\mathbb{Z}_p]\!]$. By using these facts, we can then also establish criteria in terms of explicit growth conditions that are equivalent to the limit module being finitely ∞-presented.

Assuming these growth conditions to be satisfied, we shall then prove in Theorem 6.5 (and see also the discussion following (34) in §6.2.4) that the $\mathbb{Z}[\![\mathbb{Z}_p \times H]\!]$-module arising from the cohomology in degree one of the derived inverse limit of Weil-étale cohomology complexes of $\mathbb{G}_m$ over the $\mathbb{Z}_p$-extension is both finitely-presented and torsion and that its initial Fitting ideal can be completely described in terms of the values of Dirichlet L-series. We stress here that this link between L-series and the structure of Weil-étale cohomology is established over rings $\mathbb{Z}[\![\mathbb{Z}_p \times H]\!]$ that have infinite Krull dimension and are neither coherent, compact nor regular and that it incorporates information about full divisor class groups rather than just their p-primary parts. The latter feature reflects the underlying fact that our techniques permit a unified investigation of the different ℓ-primary components of profinite Iwasawa modules, the advantages of any potential such approach having essentially already been made clear elsewhere (see, for example, the introductions to [33] and [34] and, more recently, the discussion in [3, §1.4]). A further point of interest is that the use of derived limit complexes is forced in this context, since it can be shown that the inverse limits of Weil-étale cohomology groups are themselves finitely-presented over $\mathbb{Z}[\![\mathbb{Z}_p]\!]$ if and only if the corresponding inverse systems of unit groups and norm maps are Mittag-Leffler and that this almost never happens (see Corollary 6.6).

The growth conditions on divisor class groups that are required for our approach are very mild (from the point of view, for example, of general Cohen-Lenstra-type heuristics) and, in particular, can be unconditionally verified over all constant $\mathbb{Z}_p$-extensions. This means that our theory admits further development in the latter case and the results then obtained even suggest a natural 'main conjecture of equivariant integral Iwasawa theory' for the Weil-étale cohomology of $\mathbb{G}_m$ over constant $\mathbb{Z}_p$-extensions (see, in particular, Theorem 6.11, Conjecture 6.15 and Remark 6.16).

It is, of course, widely believed, following the philosophy of Iwasawa, that constant $\mathbb{Z}_p$-extensions of global function fields of characteristic p are strongly analogous to cyclotomic $\mathbb{Z}_p$-extensions of number fields (see, for example, the discussion starting on [24, p. 188]). For this reason, the above results naturally suggest a variety of new and interesting questions over cyclotomic $\mathbb{Z}_p$-extensions of number fields concerning the structure over $\mathbb{Z}[\![\mathbb{Z}_p]\!]$ of inverse limits of both (full) ideal class groups and the Selmer groups of $\mathbb{G}_m$, and their possible consequences concerning asymptotic growth formulae. We aim to consider such issues elsewhere.

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2. Pro-discrete modules

For the reader's convenience, in this section we shall quickly review relevant aspects of the theory of pro-discrete modules developed in [7]. Throughout, we write $\mathbb{N}_0$ for the set $\mathbb{N} \cup \{0\}$ of non-negative integers and set

$$R := \mathbb{Z}[\![\mathbb{Z}_p]\!] \quad \text{and} \quad R_n := \mathbb{Z}[\mathbb{Z}/(p^n)] \text{ for } n \in \mathbb{N}_0. \tag{1}$$

We also fix a finite abelian group H and consider modules over the group ring $R[H]$. In particular, for each $R[H]$-module M and $n \in \mathbb{N}_0$ we define an $R_n[H]$-module

$$M_{(n)} := R_n \otimes_R M,$$

where the tensor product is with respect to the natural (surjective) homomorphism $R \to R_n$.

If M is a module over any ring A, then we write $\mu_A(M)$ for the cardinality of a minimal generating set, with $\mu_A(M)$ regarded as ∞ if M is not finitely-generated. For an abelian group M we write $M_{\rm tor}$ for its torsion subgroup and $M_{\rm tf}$ for the quotient of M by $M_{\rm tor}$.

We write $\mathcal{P}$ for the set of rational primes. For $\ell \in \mathcal{P}$ we write $\mathbb{F}_\ell$ for the field of cardinality ℓ and, for an abelian group M, we set $M/\ell := \mathbb{F}_\ell \otimes_{\mathbb{Z}} M$ and write $M[\ell^\infty]$ for the maximal ℓ-primary subgroup of $M_{\rm tor}$.

We first recall the definition of pro-discrete $R[H]$-module.

Definition 2.1. An inverse system $(M_n)_n = (M_n, \pi_n)_n$ of $R[H]$-modules indexed by $n \in \mathbb{N}_0$ is a *pro-discrete system* if each M_n is naturally an $R_n[H]$-module and, for each $n \in \mathbb{N}$, the transition morphism π_n induces an isomorphism $\overline{\pi}_n : (M_n)_{(n-1)} \cong M_{n-1}$ of $R_{n-1}[H]$-modules. An $R[H]$-module is *pro-discrete* if it is the limit of a pro-discrete system of $R[H]$-modules.

Remark 2.2. The ring R is itself a pro-discrete R-module since it is the limit of the pro-discrete system $(R_n, \varrho_{n,n-1})_n$ in which $\varrho_{n,n-1}$ is the natural map $R_n \to R_{n-1}$. In general, any $R[H]$-module M gives rise to a pro-discrete system $(M_{(n)}, \pi_n)_n$, with π_n the canonical map $M_{(n)} \to M_{(n-1)}$, and hence to a pro-discrete module $\varprojlim_n M_{(n)}$ for which there exists a canonical map $M \to \varprojlim_n M_{(n)}$. However, many finitely-presented $R[H]$-modules are not pro-discrete and so the category of pro-discrete $R[H]$-modules is not abelian (for more details see [7, Rem. 3.12]).

We recall that an $R_n[H]$-module M is said to be 'locally-free (of rank d)' if for every rational prime ℓ the $(\mathbb{Z}_\ell \otimes_{\mathbb{Z}} R_n)[H]$-module $\mathbb{Z}_\ell \otimes_{\mathbb{Z}} M$ is free (of rank d).

The following result provides analogues of Nakayama's Lemma and Roiter's Lemma for the category of pro-discrete modules.

Proposition 2.3 ([7, Th. 3.8])**.** *Let M be a pro-discrete $R[H]$-module that arises as the limit of a pro-discrete system $(M_n, \pi_n)_n$. Then the following claims are valid.*

(i) *$\mu_{R[H]}(M)$ is finite if and only if each $\mu_{R_n}(M_n)$ is finite and bounded independently of n.*
(ii) *If $\mu_{R[H]}(M)$ is finite, then $\mu_{\mathbb{Z}[H]}(M_0)$ is finite and there exists a pro-discrete $R[H]$-submodule M' of M with both of the following properties:*
   (a) *$\mu_{R[H]}(M') \leq \mu_{\mathbb{Z}[H]}(M_0)$;*
   (b) *For every n, the order of $(M/M')_{(n)}$ is finite and prime-to-p.*
   In particular, if each $R_n[H]$-module M_n is locally-free, then for natural numbers m and $t > 1$, the module M' can be chosen to be free and such that $\big(M/M'\big)_{(m)}$ is finite and of order prime-to-pt.

We next recall criteria for the finite-presentability of finitely-generated pro-discrete modules. To state the result we fix a topological generator γ of $\mathbb{Z}_p$ and, for each $n \in \mathbb{N}_0$, set

$$\varpi_n := \gamma^{p^n} - 1 \in R \quad \text{and} \quad \gamma_n := \varrho_n(\gamma) \in R_n,$$

where ϱ_n is the ring homomorphism $R \to R_n$ induced by the canonical projection $\mathbb{Z}_p \to \mathbb{Z}/p^n$.

Proposition 2.4 ([7, Th. 3.11])**.** *Let M be a finitely-generated $R[H]$-module.*

(i) *Then M is pro-discrete if and only if the natural map $M \to \varprojlim_n M_{(n)}$ is bijective. In particular, if M is the limit of a pro-discrete system $(M_n, \pi_n)_n$ then for every n the natural map $M_{(n)} \to M_n$ is bijective.*

(ii) *If M is pro-discrete, it is finitely-presented if and only if $\mu_{R_n}(\mathsf{Tor}_1^R(R_n, M))$ is bounded independently of n. The latter condition is satisfied if, for example, each M_n is finite and the quantities $\mu_R\big(M^{\varpi_n=0}\big)$ are finite and bounded independently of n.*

(iii) *If M is a submodule of $R[H]^a$ for some natural number a, then M is pro-discrete. In addition, any such M is finitely-presented if each group $\big(R[H]^a/M\big)^{\varpi_n=0}[p^\infty]$ has bounded exponent.*

3. Ideal theory

In this section we investigate the ideal theory of R. More precisely, we clarify aspects of the interplay between the ideal theories of R and of $R^\ell := \mathbb{Z}_\ell[\![\mathbb{Z}_p]\!]$ for each prime $\ell \in \mathcal{P}$, before proving that R has infinite Krull dimension and describing the structure of $\mathrm{MaxSpec}(R)$.

For each $n \in \mathbb{N}_0$, we write $I(n)$ for the ideal $\ker(\varrho_n)$ of R and recall from [7, Prop. 2.1(ii)] the existence of a canonical short exact sequence of R-modules

$$0 \to R \cdot \varpi_n \xrightarrow{\subseteq} I(n) \xrightarrow{\xi_n} \big(\mathbb{Z}_p/\mathbb{Z}\big) \otimes_{\mathbb{Z}} R_n \to 0, \tag{2}$$

in which R acts on the third module via $1 \otimes_R \varrho_n$ and ξ_n is a natural connecting homomorphism that arises in the theory of pro-discrete modules.

These sequences play a key role in the theory developed in [7] and, to help set context, we record the following result which shows that the classification of such sequences is not straightforward.

Lemma 3.1. *The group $\mathrm{Ext}^1_R\big((\mathbb{Z}_p/\mathbb{Z}) \otimes_{\mathbb{Z}} R_n, R \cdot \varpi_n\big)$ is isomorphic to $\mathrm{Ext}^1_R\big((\mathbb{Z}_p/\mathbb{Z}) \otimes_{\mathbb{Z}} R_n, R\big)$ and is both uniquely p-divisible and contains a rational vector space of uncountable dimension.*

Proof. The isomorphism is induced by the fact that the R-module $R \cdot \varpi_n$ is free of rank one (since R is a domain and $\varpi_n \neq 0$). To prove the remaining assertions, we set $R_n^{\langle p\rangle} := \big(\mathbb{Z}_p/\mathbb{Z}\big) \otimes_{\mathbb{Z}} R_n$ and $\mathcal{E}_n^1 := \mathrm{Ext}^1_R\big(R_n^{\langle p\rangle}, R \cdot \varpi_n\big)$. Then, as the group $\mathbb{Z}_p/\mathbb{Z}$ is uniquely p-divisible, so is $\mathcal{E}_n^1$. In addition, since $I(n)$ has no non-zero p-divisible elements, the group $\mathrm{Hom}_R\big(R_n^{\langle p\rangle}, I(n)\big)$ vanishes and so the canonical connecting homomorphism $\mathrm{Hom}_R\big(R_n^{\langle p\rangle}, R_n^{\langle p\rangle}\big) \to \mathcal{E}_n^1$ induced by (2) is injective. In particular, since the torsion subgroup of $\mathbb{Z}_p/\mathbb{Z}$ is $\mathbb{Z}_{(p)}/\mathbb{Z}$, with $\mathbb{Z}_{(p)}$ the localisation of $\mathbb{Z}$ at p, and $\mathbb{Z}_p/\mathbb{Z}_{(p)}$ is a rational vector space of uncountable dimension, it is enough to note that the group

$$\mathrm{Hom}_R\big((\mathbb{Z}_p/\mathbb{Z}_{(p)}) \otimes_{\mathbb{Z}} R_n, R_n^{\langle p\rangle}\big) \cong \mathrm{Hom}_{\mathbb{Z}}\big(\mathbb{Z}_p/\mathbb{Z}_{(p)}, R_n^{\langle p\rangle}\big) = \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Z}_p/\mathbb{Z}_{(p)}, \mathbb{Z}_p/\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} R_n)$$

is isomorphic to a subgroup of $\mathrm{Hom}_R\big(R_n^{\langle p\rangle}, R_n^{\langle p\rangle}\big)$. □

3.1. Ideals of R and of each R^ℓ.

For $\ell \in \mathcal{P}$ and $n \in \mathbb{N}$, we set $R_n^\ell := \mathbb{Z}_\ell[\![\mathbb{Z}/p^n]\!]$ and abuse notation by continuing to write ϱ_n and $\varrho_{n,n-1}$ for the homomorphisms $R^\ell \to R_n^\ell$ and $R_n^\ell \to R_{n-1}^\ell$ that are respectively induced by the canonical projections ϱ_n and $\varrho_{n,n-1}$. We also note that each ring $R^\ell = \varprojlim_{\varrho_{n,n-1}} R_n^\ell$ has a natural (compact) inverse limit topology and, for a subset X of R^ℓ, we write X^c for its closure with respect to this topology.

Proposition 3.2. *The following claims are valid for all ideals J of R.*

(i) $\bigcap_{n\in\mathbb{N}_0}(J + I(n)) = \bigcap_{\ell\in\mathcal{P}}\big(R \cap (R^\ell \cdot J)^c\big)$.

(ii) *If J is pro-discrete, then $\big(\bigcap_{n\in\mathbb{N}_0}(J + I(n))\big)^2 \subseteq J \subseteq \bigcap_{n\in\mathbb{N}_0}(J + I(n))$. In particular, if J is pro-discrete and prime, then $J = \bigcap_{\ell\in\mathcal{P}}\big(R \cap (R^\ell \cdot J)^c\big)$.*

(iii) *There exists an infinite chain $J(1) \supsetneq J(2) \supsetneq \cdots$ of principal (and hence pro-discrete) ideals of R with both of the following properties.*

(a) *For $i \in \mathbb{N}$ and $n \in \mathbb{N}_0$, one has $J(i) + I(n) = I(0)$, and hence $\varrho_n(J(i)) = R_n \cdot \varrho_n(\varpi_0)$.*
(b) *For $i \in \mathbb{N}$ and $\ell \in \mathcal{P}$, one has $R^\ell \cdot J(i) = R^\ell \cdot \varpi_0$.*

Proof. We first claim that, for each $\ell \in \mathcal{P}$ and ideal X of R^ℓ, one has

$$X^c = \varprojlim_{\varrho_{n,n-1}} \varrho_n(X). \tag{3}$$

To show this, we note that each $\varrho_n(X)$ is a $\mathbb{Z}_\ell$-submodule of R^ℓ_n, and hence compact. The limit $\varprojlim_{\varrho_{n,n-1}} \varrho_n(X)$ is therefore a compact submodule of $R^\ell = \varprojlim_{\varrho_{n,n-1}} R^\ell_n$, and hence closed. In particular, since it is clear $X \subseteq \varprojlim_{\varrho_{n,n-1}} \varrho_n(X)$, one also has $X^c \subseteq \varprojlim_{\varrho_{n,n-1}} \varrho_n(X)$. It is therefore enough to show that any closed submodule Y of R^ℓ that contains X must contain $\varprojlim_{\varrho_{n,n-1}} \varrho_n(X)$. To do this we use the obvious exact commutative diagrams

$$\begin{CD}
0 @>>> Y \cap (R^\ell \cdot \varpi_n) @>{\subseteq}>> Y @>{\varrho_n}>> \varrho_n(Y) @>>> 0\\
@. @VV{\subseteq}V @| @VV{\varrho_{n,n-1}}V @.\\
0 @>>> Y \cap (R^\ell \cdot \varpi_{n-1}) @>{\subseteq}>> Y @>{\varrho_{n-1}}>> \varrho_{n-1}(Y) @>>> 0.
\end{CD}$$

Since Y and each $R^\ell \cdot \varpi_n$ are closed submodules of R^ℓ, the first terms in both rows are compact. Recalling that limits are exact for inverse systems of compact modules (indexed by $\mathbb{N}$), it follows that $\varprojlim^1_n (Y \cap (R^\ell \cdot \varpi_n))$ vanishes. Upon passing to the inverse limit over n of the above diagrams one therefore deduces that the inclusion map $Y \to \varprojlim_{\varrho_{n,n-1}} \varrho_n(Y)$ is bijective. In particular, since $\varrho_n(X) \subseteq \varrho_n(Y)$ for each n, it follows that $\varprojlim_{\varrho_{n,n-1}} \varrho_n(X) \subseteq Y$, as required.

To prove (i) we use the fact that, for every $\ell \in \mathcal{P}$ and $n \in \mathbb{N}_0$, one has $\varrho_n(R^\ell \cdot J) = \mathbb{Z}_\ell \otimes_\mathbb{Z} \varrho_n(J)$. This implies, in particular, there exists an exact sequence

$$0 \to \varrho_n(J) \xrightarrow{\subseteq} R_n \cap \varrho_n(R^p \cdot J) \to R_n/\varrho_n(J) \xrightarrow{x \mapsto 1 \otimes x} R^p_n/\varrho_n(R^p \cdot J),$$

where the last map uses the identification $\mathbb{Z}_p \otimes_\mathbb{Z} (R_n/\varrho_n(J)) \cong R^p_n/\varrho_n(R^p \cdot J)$. In particular, since the kernel of the last map is

$$\bigoplus\nolimits_{\ell \in \mathcal{P}\setminus\{p\}} \big(R_n/\varrho_n(J)\big)[\ell^\infty] = \bigoplus\nolimits_{\ell \in \mathcal{P}\setminus\{p\}} \big(R^\ell_n/\varrho_n(R^\ell \cdot J)\big)[\ell^\infty],$$

one obtains an exact commutative diagram

$$\begin{CD}
0 @>>> \varrho_n(J) @>{\subseteq}>> R_n \cap \varrho_n(R^p \cdot J) @>{(\beta^\ell_n)_\ell}>> \bigoplus_{\ell \in \mathcal{P}\setminus\{p\}} \big(R^\ell_n/\varrho_n(R^\ell \cdot J)\big)\\
@. @VVV @VVV @VVV\\
0 @>>> \varrho_{n-1}(J) @>{\subseteq}>> R_{n-1} \cap \varrho_{n-1}(R^p \cdot J) @>{(\beta^\ell_{n-1})_\ell}>> \bigoplus_{\ell \in \mathcal{P}\setminus\{p\}} \big(R^\ell_{n-1}/\varrho_{n-1}(R^\ell \cdot J)\big)
\end{CD}$$

in which all vertical maps are induced by $\varrho_{n,n-1}$ and β^ℓ_n is induced by the inclusion $R_n \subset R^\ell_n$. By passing to the limit of these diagrams, and taking account of (3) with $X = R^p \cdot J$, we therefore obtain an exact sequence

$$0 \to \varprojlim_{\varrho_{n,n-1}} \varrho_n(J) \xrightarrow{\subseteq} R \cap (R^p \cdot J)^c \xrightarrow{(\varprojlim_n \beta^\ell_n)_\ell} \prod\nolimits_{\ell \in \mathcal{P}\setminus\{p\}} \varprojlim_{\varrho_{n,n-1}} \big(R^\ell_n/\varrho_n(R^\ell \cdot J)\big). \tag{4}$$

In addition, for each $\ell \in \mathcal{P} \setminus \{p\}$, by passing to the limit over n of the exact commutative diagrams

$$\begin{CD}
0 @>>> \varrho_n(R^\ell \cdot J) @>{\subseteq}>> R^\ell_n @>>> R^\ell_n/\varrho_n(R^\ell \cdot J) @>>> 0\\
@. @V{\varrho_{n,n-1}}VV @V{\varrho_{n,n-1}}VV @V{\varrho_{n,n-1}}VV @.\\
0 @>>> \varrho_{n-1}(R^\ell \cdot J) @>{\subseteq}>> R^\ell_{n-1} @>>> R^\ell_{n-1}/\varrho_{n-1}(R^\ell \cdot J) @>>> 0
\end{CD}$$

(in which all occurring modules are compact), and applying (3) with $X = R^\ell \cdot J$, one obtains a natural isomorphism of R^ℓ-modules $R^\ell/(R^\ell \cdot J)^c \cong \varprojlim_{\varrho_{n,n-1}} \big(R^\ell_n/\varrho_n(R^\ell \cdot J)\big)$. Upon combining these isomorphisms with (4), one obtains an exact sequence of R-modules

$$0 \to \varprojlim_{\varrho_{n,n-1}} \varrho_n(J) \xrightarrow{\subseteq} R \cap (R^p \cdot J)^c \xrightarrow{(\beta^\ell)_\ell} \prod_{\ell \in \mathcal{P}\setminus\{p\}} \big(R^\ell/(R^\ell \cdot J)^c\big)$$

in which each map β^ℓ is induced by the inclusion $R \subset R^\ell$. To deduce the equality in (i) from this exact sequence, it is then enough to note that the explicit definition of each ideal $I(n)$ as the kernel of the projection $\varrho_n : R \to R_n$ implies that $\varprojlim_{\varrho_{n,n-1}} \varrho_n(J)$ is equal to $\bigcap_{n\in\mathbb{N}_0}(J + I(n))$.

To prove (ii) we set

$$\tilde{J} := \varprojlim_{\varrho_{n,n-1}} \varrho_n(J) = \bigcap_{n\in\mathbb{N}_0} (J + I(n)) \quad \text{and} \quad Q_n[1] := \mathsf{Tor}_1^R(R/J, R_n) \quad \text{for} \quad n \in \mathbb{N}_0.$$

Then the argument of [7, Th. 3.11(i)] gives an exact commutative diagram of R_n-modules

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Q_n[1] & \longrightarrow & J_{(n)} & \longrightarrow & \varrho_n(J) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \alpha_n} & & \Big\downarrow{\scriptstyle \alpha'_n} & & \Big\downarrow{\scriptstyle \varrho_{n,n-1}} & & \\
0 & \longrightarrow & Q_{n-1}[1] & \longrightarrow & J_{(n-1)} & \longrightarrow & \varrho_{n-1}(J) & \longrightarrow & 0
\end{array}$$

in which α'_n is the canonical (surjective) map. In particular, if J is pro-discrete, then by passing to the inverse limit over n of these sequences we obtain an exact sequence of R-modules

$$0 \to J \xrightarrow{\subseteq} \tilde{J} \to \varprojlim\nolimits^1_{\alpha_n} Q_n[1] \to 0.$$

In addition, since each R_n-module $Q_n[1]$ is annihilated by $\varrho_n(J)$, the action of R on $\varprojlim^1_{\alpha_n} Q_n[1]$ factors through $R/\tilde{J}$. From the last displayed exact sequence we may therefore deduce that $\tilde{J}^2 \subseteq J$, as required to prove the first assertion of (ii). Given this, the second assertion of (ii) then follows directly from the fact that, if J is a prime ideal, then the inclusions $\tilde{J}^2 \subseteq J \subseteq \tilde{J}$ combine to imply that $J = \tilde{J}$.

To prove (iii) we fix a subset $\{n_i\}_{i\in\mathbb{N}}$ of $\mathbb{N}$ such that each n_i is prime-to-p and divides n_{i+1}, and consider the principal ideals $J(i) := R \cdot (\gamma^{n_i} - 1)$. Then, since $\gamma^{n_i} - 1$ is a divisor of $\gamma^{n_{i+1}} - 1$ in R, one has $J(i+1) \subseteq J(i)$. It is also easily checked that $\varrho_n(J(i)) = R_n \cdot (\gamma_n - 1)$ for each n and $R^\ell \cdot J(i) = R^\ell \cdot (\gamma - 1)$ for each ℓ (so that (a) and (b) are valid). It therefore suffices to prove $J(i) \neq J(j)$ for $i \neq j$ and, for this, it is enough to show that the index of $J(i)$ in $R \cdot (\gamma - 1)$ is equal to n_i. To do this, we use the exact commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R\cdot(\gamma-1) & \xrightarrow{\subset} & I(0) & \xrightarrow{\xi_0} & \mathbb{Z}_p/\mathbb{Z} & \longrightarrow & 0 \\
 & & {\scriptstyle \subseteq}\Big\uparrow & & \Big\| & & \Big\uparrow{\scriptstyle \times n_i} & & \\
0 & \longrightarrow & J(i) & \xrightarrow{\subset} & I(0) & \xrightarrow{\xi_{0,i}} & \mathbb{Z}_p/\mathbb{Z} & \longrightarrow & 0.
\end{array} \tag{5}$$

Here the upper row is (2) with $n = 0$ and the lower row is the exact sequence constructed by the argument of [7, Prop. 2.1(ii)] after replacing $\gamma - 1$ by $\gamma^{n_i} - 1 = (\sum_{a=0}^{a=n_i-1} \gamma^a)(\gamma - 1)$. The commutativity of the first square is then clear and that of the second square is checked by explicitly comparing the respective connecting homomorphisms ξ_0 and $\xi_{0,i}$. Hence, since the kernel of multiplication by n_i on $\mathbb{Z}_p/\mathbb{Z}$ is cyclic of order n_i, the Snake Lemma applies to the above diagram to imply that the index of $J(i)$ in $R \cdot (\gamma - 1)$ is n_i, as claimed. □

Remark 3.3. (i) A finitely generated ideal X of R^ℓ is compact, and hence closed, so that $X = X^c$. In particular, since R^p is Noetherian, all of its ideals are closed. If $\ell \neq p$, then Bandini and Longhi have shown that an ideal of R^ℓ is closed if and only if it is principal (see [3, Prop. 2.3]).

(ii) A finitely generated ideal of R is pro-discrete (by Proposition 2.4(iii)), but not all pro-discrete ideals are finitely generated. For example, $I(0)$ is pro-discrete (by [7, Lem. 3.6]) but the surjectivity of the map ξ_0 in (2) implies $I(0)$ cannot be generated by any countable set of elements.
(iii) Proposition 3.2 (ii) and (iii) combine to imply the ideals $J(i)$ of R are not prime, and this also follows from the fact the lower row in (5) implies $I(0)^2 \subseteq J(i)$ and $I(0) \not\subseteq J(i)$. Whilst these ideals show that even principal ideals of R are not determined by their extensions to R^ℓ for all $\ell \in \mathcal{P}$, they are still distinguishable in these rings since, if $i < j$, then for any prime divisor ℓ of n_j/n_i the images in R^ℓ of the completions $\varprojlim_n (J(i)/\ell^n)$ and $\varprojlim_n (J(j)/\ell^n)$ differ. In particular, in relation to later arithmetic questions, it would be interesting to know if finitely-generated (or perhaps pro-discrete) ideals J of R are determined by knowledge of both $\varrho_n(J)$ for all $n \in \mathbb{N}$ and $\varprojlim_n (J/\ell^n)$ for all $\ell \in \mathcal{P}$.

We next show that an ideal of the classical Iwasawa algebra $R^p = \mathbb{Z}_p[[\mathbb{Z}_p]]$ is uniquely determined by its restriction to R.

Proposition 3.4. *For every ideal I of R^p, one has $R^p \cdot (I \cap R) = I$.*

Proof. We can fix $n \in \mathbb{N}$ and a subset $\{f_i\}_{1 \le i \le n}$ of I such that $I = \sum_{i=1}^{i=n} R^p \cdot f_i$. Then, assuming that $R^p \cdot ((R^p \cdot f_i) \cap R) = R^p \cdot f_i$ for every i, one has

$$I = \sum_{i=1}^{i=n} R^p \cdot f_i = \sum_{i=1}^{i=n} R^p \cdot ((R^p \cdot f_i) \cap R) \subseteq R^p \cdot \sum_{i=1}^{i=n} ((R^p \cdot f_i) \cap R) \subseteq R^p \cdot (I \cap R) \subseteq I$$

and hence $R^p \cdot (I \cap R) = I$. It is therefore enough to prove the stated equality in the case that $I = R^p \cdot f$ for some $f \in R^p$.

For $i \in \mathbb{N}_0$, we write $\Phi_i(X)$ for the p^i-th cyclotomic polynomial in $\mathbb{Z}[X]$. We recall that $\Phi_i(X)$ is irreducible as an element of the polynomial ring $\mathbb{Z}_p[X]$ and further that, for each $n \in \mathbb{N}$, one has $X^{p^n} - 1 = \prod_{i=0}^{i=n} \Phi_i(X)$ in $\mathbb{Z}[X]$. Now if, for any $m \in \mathbb{N}_0$, the quotient of R^p by the ideal $R^p \cdot f + R^p \cdot \varpi_m$ is infinite, then f must be divisible in R^p by $\Phi_i(\gamma)$ for some i with $0 \le i \le m$. Hence, after dividing f by a finite (possibly empty) product g of elements of the form $\Phi_i(\gamma)$, we obtain an ideal $I' := R^p \cdot (fg^{-1})$ with the property that $\varrho_n(I')$ has finite index in R_n^p for every $n \in \mathbb{N}_0$. In addition, as g belongs to R and is a non-zero divisor in R^p, one has $R^p \cdot ((R^p \cdot f) \cap R) = R^p \cdot f$ if $R^p \cdot ((R^p \cdot (fg^{-1})) \cap R) = R^p \cdot fg^{-1}$. It follows that, after replacing f by fg^{-1} if necessary, we can in the sequel assume that $R_n^p/\varrho_n(I)$ is finite for every $n \in \mathbb{N}_0$.

In this case, for each n one has $R_n + \varrho_n(I) = R_n^p$ and hence also $R_n^p \cdot (R_n \cap \varrho_n(I)) = \varrho_n(I)$. This in turn gives an exact commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R_n \cap \varrho_n(I) & \xrightarrow{x \to (x,x)} & R_n \oplus \varrho_n(I) & \xrightarrow{(x,y) \mapsto x-y} & R_n^p & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_n} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R_{n-1} \cap \varrho_{n-1}(I) & \xrightarrow{x \to (x,x)} & R_{n-1} \oplus \varrho_{n-1}(I) & \xrightarrow{(x,y) \mapsto x-y} & R_{n-1}^p & \longrightarrow & 0
\end{array} \tag{6}$$

in which every vertical map is induced by $\varrho_{n,n-1}$. In addition, since $\varrho_n(I)$ is a free R_n^p-module of rank one (under the current hypotheses), the natural map $(\varrho_n(I))_{(n-1)} \to \varrho_{n-1}(I)$ is bijective. Using this last fact, an easy diagram chase implies that the map α_n is surjective.

Now, by using (3) (and Remark 3.3(i)) with $X = I$, the inverse limit over n of the above diagrams gives an exact sequence

$$0 \to \varprojlim_{\varrho_{n,n-1}} (R_n \cap \varrho_n(I)) \xrightarrow{x \to (x,x)} R \oplus I \xrightarrow{(x,y) \mapsto x-y} R^p,$$

and thereby an equality $R \cap I = \varprojlim_{\varrho_{n,n-1}} (R_n \cap \varrho_n(I))$. For each m, it then follows that

$$\varrho_m\big(R^p \cdot (R \cap I)\big) = R_m^p \cdot \varrho_m\big(\varprojlim_{\varrho_{n,n-1}} (R_n \cap \varrho_n(I))\big) = R_m^p \cdot (R_m \cap \varrho_m(I)) = \varrho_m(I),$$

where the second equality follows from the surjectivity of each map α_n and the third was observed just before (6).

Hence, since (3) (and Remark 3.3(i)) implies $R^p\cdot(R\cap I)=\varprojlim_m \varrho_m\big(R^p\cdot(R\cap I)\big)$, it therefore follows that $R^p\cdot(I\cap R)=I$, as claimed. $\square$

3.2. **Observations on** $\mathrm{Spec}(R)$**.** In this subsection, we make several observatons about prime ideals of R. As a first step, we determine all relevant Krull dimensions.

Proposition 3.5.

(i) $\dim(R^p)=2$ *and* $\dim(R/p)=1$.
(ii) $\dim(R^\ell)=\infty$ *and* $\dim(R/\ell)=0$ *for* $\ell\in\mathcal{P}\setminus\{p\}$.
(iii) $\dim(R)=\infty$.

Proof. For $\ell\in\mathcal{P}$ one has $R^\ell=\mathbb{Z}_\ell[\![\mathbb{Z}_p]\!]$ and $R/\ell=\mathbb{F}_\ell[\![\mathbb{Z}_p]\!]$. Claim (i) is therefore a result of classical Iwasawa theory, but to prove (ii) and (iii) we shall use results of Maroscia and of Gilmer and Heinzer. For this, we write $\mathcal{O}_m$ for $m\in\mathbb{N}_0$ for the ring of algebraic integers of $\mathbb{Q}(e^{2\pi i/p^m})$ and, for each $n\geq m$, we write $\tau_{n,m}$ for the $\mathbb{Z}[1/p]$-linear ring homomorphism $R_n[1/p]\to\mathcal{O}_m[1/p]$ that sends γ_n to $e^{2\pi i/p^m}$. Then, for each $n\in\mathbb{N}$, there is a canonical isomorphism of $\mathbb{Z}[1/p]$-algebras

$$R_n[1/p]\cong\prod\nolimits_{m\in\{0\}\cup[n]}\mathcal{O}_m[1/p],\quad x\mapsto(\tau_{n,m}(x))_m. \tag{7}$$

In particular, if we fix $\ell\in\mathcal{P}\setminus\{p\}$, then these decompositions combine to induce isomorphisms of $\mathbb{Z}_\ell$-algebras

$$R^\ell\cong\prod\nolimits_{m\in\mathbb{N}_0}(\mathbb{Z}_\ell\otimes_{\mathbb{Z}}\mathcal{O}_m)\quad\text{and}\quad R^\ell/\ell\cong R/\ell\cong\prod\nolimits_{m\in\mathbb{N}_0}\big(\mathcal{O}_m/\ell\big). \tag{8}$$

Now, since $\dim(\mathbb{Z}_\ell\otimes_{\mathbb{Z}}\mathcal{O}_m)=1$ for every m, the first isomorphism combines with a result [14, Th. 3.3] of Gilmer and Heinzer to imply $\dim(R^\ell)=\infty$. On the other hand, since the Chinese Remainder Theorem implies each ring $\mathcal{O}_m/\ell$ is a finite direct product of fields (as $\ell\neq p$), the second isomorphism combines with a result [22, Prop. 2.6] of Maroscia to imply $\dim(R/\ell)=0$. This proves (ii).

To prove (iii), we write $R(\ell)$ for the direct product ring $\prod_{m\in\mathbb{N}_0}\big(\mathcal{O}_m/\ell^m\big)$. Then, since the image of ℓ in each ring $\mathcal{O}_m/\ell^m$ is nilpotent, but the image of ℓ under the diagonal map $R\to R(\ell)$ is not nilpotent, the result of [14, Th. 3.4] implies that $\dim(R(\ell))=\infty$. To prove (iii), it is therefore enough to construct a surjective ring homomorphism from R to $R(\ell)$. To do this, for each $n\in\mathbb{N}_0$ we write $R(\ell;n)$ for the ring $\bigoplus_{m\in\{0\}\cup[n]}\big(\mathcal{O}_m/\ell^m\big)$ and consider the composite ring homomorphism

$$\pi_n: R_n\subset R_n[1/p]\cong\prod\nolimits_{m\in\{0\}\cup[n]}\mathcal{O}_m[1/p]\to\prod\nolimits_{m\in\{0\}\cup[n]}\big(\mathcal{O}_m[1/p]\big)/\ell^m=R(\ell;n), \tag{9}$$

where the isomorphism is as in (7), the second arrow is the natural projection and the equality is valid since $\ell\neq p$. For each $n\in\mathbb{N}$, one then has an exact commutative diagram of R-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker(\pi_n) & \xrightarrow{\subseteq} & R_n & \xrightarrow{\pi_n} & R(\ell;n) & \longrightarrow & 0\\ & & \Big\downarrow{\scriptstyle\varrho'_{n,n-1}} & & \Big\downarrow{\scriptstyle\varrho_{n,n-1}} & & \Big\downarrow{\scriptstyle\rho_{n,n-1}} & & \\ 0 & \longrightarrow & \ker(\pi_{n-1}) & \xrightarrow{\subseteq} & R_{n-1} & \xrightarrow{\pi_{n-1}} & R(\ell,n-1) & \longrightarrow & 0 \end{array} \tag{10}$$

in which $\varrho'_{n,n-1}$ denotes the restriction of $\varrho_{n,n-1}$ and $\rho_{n,n-1}$ the natural projection map. It is then enough to show the maps $\varrho'_{n,n-1}$ are surjective since, if this is true, upon passing to the inverse limit over n of these diagrams one would deduce that the induced composite map

$$R=\varprojlim\nolimits_{\varrho_{n,n-1}}R_n\xrightarrow{(\pi_n)_n}\varprojlim\nolimits_{\rho_{n,n-1}}R(\ell;n)=R(\ell)$$

is a surjective ring homomorphism, as required. Now, as $\varrho_{n,n-1}$ is surjective, the Snake Lemma implies $\varrho'_{n,n-1}$ is surjective if $\pi_n(\ker(\varrho_{n,n-1}))=\ker(\rho_{n,n-1})$. To check this we write e_1 for the idempotent $(1/p)\sum_{i\in[p]}\gamma_n^{p^{n-1}i}$ of $R_n[1/p]$. Then $\tau_{n,n}((1-e_1))=1\in\mathcal{O}_n$ and $p(1-e_1)$ belongs to $R_n\cdot(\gamma_n^{p^{n-1}}-1)=\ker(\varrho_{n,n-1})$. Hence, since $\ker(\rho_{n,n-1})=\mathcal{O}_n/\ell^n$ and $\ell\neq p$, it follows that $\pi_n(\ker(\varrho_{n,n-1}))=\ker(\rho_{n,n-1})$, as required to complete the proof. □

Remark 3.6. The proof of Proposition 3.5(ii) shows that, contrary to the result of Proposition 3.4, for each $\ell\in\mathcal{P}\setminus\{p\}$ there are infinitely many prime ideals of R^ℓ whose intersection with R is (0).

Whilst Proposition 3.5 implies that obtaining a fully explicit description of $\mathrm{Spec}(R)$ is difficult, it is still possible to make several general observations about prime ideals.

For example, if J is a prime ideal that contains ϖ_m for some $m\in\mathbb{N}_0$, then since the exact sequence (2) implies $I(m)^2\subseteq R\cdot\varpi_m$, one has $I(m)^2\subseteq J$ and hence also $I(m)\subseteq J$. It follows that $J=\bigcap_{\ell\in\mathcal{P}}\big(R\cap(R^\ell\cdot J)^c\big)$ (by Proposition 3.2(i)) and also that J corresponds to a prime ideal of R_m. From [7, Th. 4.2(iii)], one then also knows that J is finitely generated if and only if it is a maximal ideal (that, necessarily, contains a rational prime).

Next we note that finitely generated prime ideals J that are disjoint from $\{\varpi_n\}_{n\in\mathbb{N}_0}$ correspond bijectively to irreducible distinguished polynomials in $\mathbb{Z}_p[\![X]\!]$ that are not equal to any $\Phi_i(X)$. Indeed, by [7, Th. 4.2(iv)], one knows that, for any such J, there exists a polynomial f_J of the stated kind with $J=R\cap(R^p\cdot f_J(\gamma))$ and then f_J is unique since Proposition 3.4 implies $R^p\cdot f_J=R^p\cdot J$.

To describe $\mathrm{Spec}(R)$ fully, it therefore only remains to characterise non-finitely-generated prime ideals of R that are disjoint from $\{\varpi_n\}_{n\in\mathbb{N}_0}$, and, for the most part, these remain mysterious. However, in the case of maximal ideals, further analysis is possible. To state a result in this direction, we note the maximum spectrum $\mathrm{MaxSpec}(R)$ of R decomposes as a disjoint union, parametrised by ideals $\mathfrak{a}\in\mathrm{Spec}(\mathbb{Z})$, of subspaces $\mathrm{MaxSpec}_{\mathfrak{a}}(R)$ corresponding to those maximal ideals whose residue characteristic generates $\mathfrak{a}$. The following result (some of which is well-known) describes these individual subspaces and, in particular, obtains a complete description of the set of pro-discrete maximal ideals, thereby extending the classification of finitely-generated prime ideals of R given by [7, Th. 4.2].

Proposition 3.7. *The following claims are valid.*

(i) $\mathrm{MaxSpec}_{(p)}(R)$ *is a singleton comprising the ideal generated by* $\{p,\varpi_0\}$ *which is pro-discrete and has residue field* $\mathbb{F}_p$.

(ii) $\mathrm{MaxSpec}_{(\ell)}(R)$ *for* $\ell\in\mathcal{P}\setminus\{p\}$ *is homeomorphic to the Stone-Čech compactification of* $\mathbb{N}$ *and is compact, Hausdorff and non-Noetherian. In addition, the following conditions on an ideal* $\mathfrak{m}$ *in* $\mathrm{MaxSpec}_{(\ell)}(R)$ *are equivalent (and imply its residue field is finite):* $\mathfrak{m}$ *is pro-discrete;* $\mathfrak{m}$ *is finitely-generated;* $\mathfrak{m}$ *is principal;* $\mathfrak{m}\cap\{\varpi_n\}_{n\in\mathbb{N}_0}\neq\emptyset$; $(R^\ell\cdot\mathfrak{m})^c\neq R^\ell$.

(iii) *If* $\mathfrak{m}$ *belongs to* $\mathrm{MaxSpec}_{(0)}(R)$, *then* $\mathfrak{m}$ *is not pro-discrete and* $(R^\ell\cdot\mathfrak{m})^c=R^\ell$ *for all* $\ell\in\mathcal{P}$. *In addition, the residue fields of infinitely many ideals in* $\mathrm{MaxSpec}_{(0)}(R)$ *are of uncountable dimension over* $\mathbb{Q}$.

Proof. The unique maximal ideal of $R/pR\cong\mathbb{F}_p[\![\mathbb{Z}_p]\!]$ is generated by the iamge of ϖ_0 and has residue field $\mathbb{F}_p$. The ideal generated by $\{p,\varpi_0\}$ is therefore the unique element of $\mathrm{MaxSpec}_{(p)}(R)$ and is automatically pro-discrete (by Proposition 2.4(iii)) since it is finitely-generated. This proves (i).

To prove (ii) we fix $\ell\in\mathcal{P}\setminus\{p\}$ and note the second isomorphism in (8) implies points in $\mathrm{MaxSpec}_{(\ell)}(R)$ correspond to maximal ideals of a ring $\prod_{m\in\mathbb{N}_0}\big(\mathcal{O}_m/\ell\big)$ that is isomorphic to an infinite direct product of (finite) fields. Such maximal ideals therefore correspond bijectively with the set of ultrafilters on $\mathbb{N}$ and, except perhaps for the assertion of non-Noetherianity, the first sentence of claim (ii) is a well-known consequence of this fact (see, for example, [28] and the references contained therein). In addition, every ideal of R/ℓ is radical (by [28, Th. 3.1(iii)]) and every prime ideal of

R/ℓ is maximal (by Proposition 3.5(ii)) and so the non-Noetherianity of the space $\mathrm{MaxSpec}_{(\ell)}(R)$ follows directly from that of the ring R/ℓ.

To prove the remainder of (ii) we note first that [28, Th. 3.1(i)] implies a prime ideal $\mathfrak{m}$ of R/ℓ is finitely-generated if and only if it is principal (and thereby corresponds to a principal ultrafilter), in which case $R/\mathfrak{m}$ is finite and so $\mathfrak{m}$ contains ϖ_n for all sufficiently large n. Conversely, if $\mathfrak{m}$ contains ϖ_n for any given n, then (as observed just after the proof of Proposition 3.5) $\mathfrak{m}$ is the full pre-image of a maximal ideal of R_n/ℓ and so is finitely-generated since the kernel of the projection $R/\ell \to R_n/\ell$ is generated by the image of ϖ_n. Next we show that $\mathfrak{m}$ is pro-discrete if and only if it is finitely-generated. Taking account of Proposition 2.4(iii), it is enough for this to assume $\mathfrak{m}$ is pro-discrete, and prove $\overline{\mathfrak{m}} := \mathfrak{m}/(\ell)$ is finitely-generated. To do this, we note that the tautological exact sequence $0 \to (\ell) \xrightarrow{\iota} \mathfrak{m} \to \overline{\mathfrak{m}} \to 0$ induces, upon passing to coinvariants, a family of exact commutative diagrams of R_n-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\ell)_{(n)} & \xrightarrow{\iota_{(n)}} & \mathfrak{m}_{(n)} & \longrightarrow & \overline{\mathfrak{m}}_{(n)} & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & (\ell)_{(n-1)} & \xrightarrow{\iota_{(n-1)}} & \mathfrak{m}_{(n-1)} & \longrightarrow & \overline{\mathfrak{m}}_{(n-1)} & \longrightarrow & 0. \end{array}$$

Here all vertical maps are the natural projections and injectivity of each $\iota_{(n)}$ follows from the fact $(\ell)_{(n)} \cong R_n$ is torsion-free whilst $\mathrm{Tor}_1^R(R_n, \overline{\mathfrak{m}})$ is annihilated by ℓ. Since the first vertical maps in the above diagrams are surjective, exactness is preserved upon passing to the inverse limit, thus giving (as both $(\ell) \cong R$ and $\mathfrak{m}$ are pro-discrete) an exact sequence $0 \to (\ell) \xrightarrow{\iota} \mathfrak{m} \to \varprojlim_n \overline{\mathfrak{m}}_{(n)} \to 0$. Upon comparing this sequence with the original, one deduces that the natural map $\overline{\mathfrak{m}} \to \varprojlim_n \overline{\mathfrak{m}}_{(n)}$ is bijective. Since $\overline{\mathfrak{m}}_{(n)}$ identifies with the image of $\overline{\mathfrak{m}}$ under the projection $R/\ell \to \prod_{m \in \{0\}\cup[n]}(\mathcal{O}_m/\ell)$ induced by (7), one therefore has $\overline{\mathfrak{m}} = \prod_{m\in\mathbb{N}_0} \mathrm{im}(\kappa_m)$, with κ_m the map $\overline{\mathfrak{m}} \subset R/\ell \to \mathcal{O}_m/\ell$ induced by $\tau_{n,m}$. The fact that $\overline{\mathfrak{m}}$ is a maximal ideal of R/ℓ now implies that $\mathrm{im}(\kappa_m) = \mathcal{O}_m/\ell$ for all but one value of m, and it is then clear that $\prod_{m\in\mathbb{N}_0} \mathrm{im}(\kappa_m)$ is finitely-generated over R/ℓ, as required. To complete the proof of (ii) it is now enough to prove that $\mathfrak{m} \cap \{\varpi_n\}_{n\in\mathbb{N}_0} = \emptyset \iff (R^\ell \cdot \mathfrak{m})^c = R^\ell$. The forward implication here is valid since if $\mathfrak{m} \cap \{\varpi_n\}_{n\in\mathbb{N}_0} = \emptyset$, then the maximality of $\mathfrak{m}$ implies $\mathfrak{m} + I(n) = R$ for all n and hence that $(R^\ell \cdot \mathfrak{m})^c = R^\ell$ as a consequence of Proposition 3.2(i). On the other hand, the converse implication follows from the fact that, for any n, every ideal of the ring R_n/ℓ is closed for the ℓ-adic topology.

To prove (iii) we fix $\mathfrak{m}$ in $\mathrm{MaxSpec}_{(0)}(R)$. Then, since no ring R_n can have a non-zero uniquely divisible quotient, one has $\mathfrak{m} \cap \{\varpi_i\}_{i\in\mathbb{N}_0} = \emptyset$ and, just as above, this implies $\mathfrak{m} + I(n) = R$ for every n. From the first assertion of Proposition 3.2(ii) it now follows that $\mathfrak{m}$ is not pro-discrete, whilst Proposition 3.2(i) implies that $(R^\ell \cdot \mathfrak{m})^c = R^\ell$ for every $\ell \in \mathcal{P}$.

To complete the proof of (iii), we shall now show that, for each integer $a > 1$, there exists a maximal ideal $\mathfrak{m}_a$ of R (with $\mathfrak{m}_a \neq \mathfrak{m}_{a'}$ if $a \neq a'$) that contains $\gamma - a$ and is such that $R/\mathfrak{m}_a$ has characteristic zero and uncountably infinite dimension over $\mathbb{Q}$. To do this, we set $X(a) := R\cdot(\gamma - a)$, $X(a)_n := \varrho_n(X(a))$, $X(a)^\ell = R^\ell \cdot X(a)$ and $X(a)^\ell_n = \varrho_n(R^\ell \cdot X(a))$ for $n \in \mathbb{N}_0$ and $\ell \in \mathcal{P}$. Then, since $R_n/X(a)_n$ is the cokernel of the endomorphism of R_n given by multiplication by $\varrho_n(\gamma) - a$, a simple determinant computation shows $R_n/X(a)_n$ to be finite and of order $a^{p^n} - 1$. It follows that

$$R/X(a) = \varprojlim_n (R_n/X(a)_n) = \varprojlim_n \Big(\bigoplus_{\ell\in\mathcal{P}} (R_n^\ell/X(a)_n^\ell)\Big) = \prod_{\ell\in\mathcal{P}} \varprojlim_n \big(R_n^\ell/X(a)_n^\ell\big) = \prod_{\ell\in\mathcal{P}(a)} N^\ell(a).$$

Here all inverse limits are with respect to the maps induced by the natural projections $\varrho_{n,n-1}$, so that the first equality is valid since $\varrho_{n,n-1}(X(a)_n) = X(a)_{n-1}$ for all $n \in \mathbb{N}$. In addition, we have set

$$\mathcal{P}(a) := \{\ell \in \mathcal{P} : \exists\, n \in \mathbb{N}_0,\, \ell \mid (a^{p^n} - 1)\}$$

and $N^\ell(a) := R^\ell/X(a)^\ell$ for $\ell\in\mathcal{P}$. In particular, since (3) implies $X(a)^\ell = \varprojlim_{\varrho_{n,n-1}} X(a)^\ell_n$, and also $\varrho_{n,n-1}(X(a)^\ell_n) = X(a)^\ell_{n-1}$ for $n\in\mathbb{N}$, one has $N^\ell(a) = \varprojlim_n\bigl(R^\ell_n/X(a)^\ell_n\bigr) = \varprojlim_n\bigl(\mathbb{Z}_\ell\otimes_{\mathbb{Z}}(R_n/X(a)_n)\bigr)$ and so $N^\ell(a)\neq(0)$ if and only if $\ell\in\mathcal{P}(a)$.

Now, as a special case of Zsigmondy's Theorem [35], the set $\mathcal{P}(a)$ is infinite. It follows that $\bigoplus_{\ell\in\mathcal{P}(a)}N^\ell(a)$ is a proper ideal of $R/X(a)$ and so we can choose a maximal ideal $\overline{\mathfrak{m}}_a$ of $R/X(a)$ containing $\bigoplus_{\ell\in\mathcal{P}(a)}N^\ell(a)$, and then write $\mathfrak{m}_a$ for the maximal ideal of R given by the full pre-image of $\overline{\mathfrak{m}}_a$. To show that the field $R/\mathfrak{m}_a$ has characteristic 0, we argue by contradiction and so assume it is annihilated by multiplication by ℓ_0 for some $\ell_0\in\mathcal{P}$. Then, since multiplication by ℓ_0 acts invertibly on $\prod_{\ell\in\mathcal{P}(a)\setminus\{\ell_0\}}N^\ell(a)$, one has $\prod_{\ell\in\mathcal{P}(a)\setminus\{\ell_0\}}N^\ell(a)\subseteq\overline{\mathfrak{m}}_a$ and hence also

$$R/X(a) = N^{\ell_0}(a)\oplus\prod\nolimits_{\ell\in\mathcal{P}(a)\setminus\{\ell_0\}}N^\ell(a)\subseteq\overline{\mathfrak{m}}_a.$$

This contradicts the fact that $\overline{\mathfrak{m}}_a$ is a proper ideal of $R/X(a)$ and so $R/\mathfrak{m}_a$ must have characteristic 0, as claimed.

To prove $\dim_{\mathbb{Q}}(R/\mathfrak{m}_a)$ is uncountably infinite, it is enough to show that $R/\mathfrak{m}_a$ is uncountable, and to do this we label the elements of $\mathcal{P}(a)$ in ascending order as $\{p_i\}_{i\in\mathbb{N}}$. It is then enough to note that the function $\{0,1\}^{\mathbb{N}}\to R/\mathfrak{m}_a$ sending each f to the image of $(\sum_{n=1}^{n=m(k)}f(n)2^{n-1})_k\in\prod_k R^{p_k}$ in $R/\mathfrak{m}_a = \bigl(\prod_k(R^{p_k}/X(a)^{p_k})\bigr)/\overline{\mathfrak{m}}_a$ is injective, where each $m(k)\in\mathbb{N}$ is defined via the inequalities $2^{m(k)}-1<p_k\le 2^{m(k)+1}-1$. (This argument is motivated by the StackExchange discussion [25].)

Finally, we note that if $a>a'$ and $\mathfrak{m}_a=\mathfrak{m}_{a'}$, then $\mathfrak{m}_a$ contains $a-a'=(\gamma-a)-(\gamma-a')$. This would imply the field $R/\mathfrak{m}_a$ is annihilated by multiplication by $a-a'$ and this contradicts the fact it has characteristic 0. □

Remark 3.8. (i) The proof of Proposition 3.7(iii) shows that, for an ideal $\mathfrak{m}$ in $\mathrm{MaxSpec}_{(0)}(R)$, one has $\varrho_n(\mathfrak{m})=R_n$ for all $n\in\mathbb{N}_0$ and, since $R/\mathfrak{m}$ is uniquely divisible, also $\varprojlim_n(\mathfrak{m}/\ell^n)=R^\ell$ for all $\ell\in\mathcal{P}$. It follows that the question raised at the end of Remark 3.3(iii) has a negative answer if one omits the hypothesis that J is pro-discrete.
(ii) Set $I:=I(0)$, write J for the sub-ideal of I generated by $\{p\varpi_0\}\cup\{\varpi_n-p\varpi_{n+1}\}_{n\in\mathbb{N}_0}$ and let π_J denote the projection $R\to R/J$. Then one has $\mathrm{Spec}(R/J)=\{\pi_J(I)\}\cup\{\pi_J(I+R\cdot\ell)\}_{\ell\in\mathcal{P}}$ so that $\dim(R/J)=1$. In addition, the upper row of (5) combines with the fact $\pi_J(\varpi_n)=p\cdot\pi_J(\varpi_{n+1})$ has order p^{n+1} to imply that R/J has a subgroup isomorphic to $\mathbb{Q}/\mathbb{Z}$, thereby complementing the uniquely divisible quotients of R arising from Proposition 3.7(iii).

4. Module theory

In this section, we extend, or refine, several of the general results about pro-discrete modules that are obtained in [7].

4.1. **Projective modules.**

We first describe a criterion for a pro-discrete $\mathbb{Z}[\![\mathbb{Z}_p]\!][H]$-module to be projective. Before stating it, we recall that a prime p is said-to-be 'exceptional' in [7] if it is irregular, validates Vandiver's Conjecture and is such that the p-adic λ-invariant of every (odd) isotypic factor of the ideal class group of $\mathbb{Q}(e^{2\pi i/p})$ is at least $p-1$. We further recall that heuristic arguments suggest there should be only finitely many exceptional p [20, Chap. 10, App.] and that, in any case, results of Hart et al [18] imply such a prime must be greater than 2^{31}.

Proposition 4.1. *Set $R=\mathbb{Z}[\![\mathbb{Z}_p]\!]$ and let $M=\varprojlim_n M_n$ be a finitely-generated pro-discrete $R[H]$-module. Then, unless p is exceptional, the $R[H]$-module M is projective if and only if, for every n, the $R_n[H]$-module M_n is projective.*

Proof. It is enough to show that, if p is not exceptional, then M is a projective $R[H]$-module if every M_n is a projective $R_n[H]$-module. To do this, we set $\mathbb{A} := R[H]$ and $\mathbb{A}_n := R_n[H]$ for each $n \in \mathbb{N}_0$.

We first aim to prove that, for each ℓ, the $\mathbb{A}/\ell$-module M/ℓ is free. To do this we note that, since each M_n is torsion-free, one has $M/\ell = \varprojlim_n (M_n/\ell)$ and we further recall that, by Swan's Theorem (cf. [13, Th. 32.11]), each $\mathbb{A}_n$-module M_n is both locally-free and of rank independent of n since M_{n-1} is isomorphic to $(M_n)_{(n-1)}$.

To deal with the case $\ell = p$, we note the final assertion of Proposition 2.4(i) implies that the $\mathbb{F}_p[H]$-module $(M/p)_0 := \mathbb{F}_p[H] \otimes_{\mathbb{A}} (M/p)$ is equal to M_0/p. In particular, since $\mathbb{Z}_p \otimes_{\mathbb{Z}} M_0$ is a free $\mathbb{Z}_p[H]$-module, the $\mathbb{F}_p[H]$-module $M_0/p = (\mathbb{Z}_p \otimes_{\mathbb{Z}} M_0)/p$ is free, and we write t for its rank. By applying Nakayama's Lemma for the Noetherian ring $\mathbb{A}/p$, we can thus deduce that M/p is generated as a $\mathbb{A}$-module by any set $\mathfrak{b} := \{b_i\}_{1\le i\le t}$ that maps to a basis of $(M/p)_0$. It follows that $\mathfrak{b}' = \bigcup_{h\in H}\{h(b_i)\}_{1\le i\le t}$ is a minimal set of generators of M/p over the principal ideal domain R/p. In addition, since the assumption that p is not exceptional combines with [7, Th. 3.8(iii)(a)] to imply M is a projective R-module, its quotient M/p is a projective, and hence free, module over R/p and so $\mathfrak{b}'$ must be a basis. It follows that M/p is a free $\mathbb{A}/p$-module (with basis $\mathfrak{b}$), as claimed.

We next assume $\ell \neq p$. In this case, the second isomorphism in (8) induces algebra decompositions $\mathbb{A}/\ell \cong \prod_{m\in\mathbb{N}_0}(\mathcal{O}_m[H]/\ell)$ and $\mathbb{A}_n/\ell \cong \prod_{m=0}^{m=n}(\mathcal{O}_m[H]/\ell)$ for each $n \in \mathbb{N}_0$. In particular, since each $\mathbb{A}_n/\ell$-module $M_n/\ell = (\mathbb{Z}_\ell \otimes_{\mathbb{Z}} M_n)/\ell$ is free, and of rank that is independent of n, it is straightforward to check directly that M/ℓ is a free $\mathbb{A}/\ell$-module, as required.

At this point, since M is finitely generated over $\mathbb{A}$, we can fix an exact sequence of $\mathbb{A}$-modules $0 \to K \to F \to M \to 0$ in which F is free of finite rank. We set $X := \mathrm{Hom}_R(M, K)$ and note that, since M is a projective R-module (as observed earlier), the induced sequence

$$0 \to X \to \mathrm{Hom}_R(M, F) \to \mathrm{Hom}_R(M, M) \to 0 \tag{11}$$

is exact. In addition, since K is torsion-free, for each ℓ the exact sequence $0 \to K \xrightarrow{\ell} K \to K/\ell \to 0$ induces an isomorphism of $R[H]$-modules

$$X/\ell \cong \mathrm{Hom}_R(M, K/\ell) \cong \mathrm{Hom}_{R/\ell}(M/\ell, K/\ell).$$

Hence, since M/ℓ is a free $\mathbb{A}/\ell$-module, and thus induced as an H-module, the H-module X/ℓ is also induced (cf. the proof of [2, §9, Cor.]) and hence cohomologically-trivial. In particular, by considering the exact sequence of Tate cohomology of the exact sequence $0 \to X \xrightarrow{\ell} X \to X/\ell \to 0$ we can now deduce that, for each integer q and subgroup J of the Sylow ℓ-subgroup $H(\ell)$ of H, the map $\hat{H}^q(J, X) \xrightarrow{\ell} \hat{H}^q(J, X)$ is bijective. Since each $\hat{H}^q(J, X)$ is an ℓ-group, it therefore follows that X is a cohomologically-trivial $H(\ell)$-module. Since this is true for all ℓ, it then follows that X is a cohomologically-trivial H-module (cf. [2, §9, Th. 9]).

In particular, by taking H-invariants of the exact sequence (11), we now deduce that the map $\mathrm{Hom}_{\mathbb{A}}(M, F) \to \mathrm{Hom}_{\mathbb{A}}(M, M)$ is surjective and hence that the identity map of M extends to a map $M \to F$ of $\mathbb{A}$-modules. It follows that M is a direct summand of the (free) $\mathbb{A}$-module F and so is projective, as required. □

Remark 4.2. Whilst the argument used to prove Proposition 4.1 relies on the hypothesis that p is not exceptional (via an application of [7, Th. 3.8(iii)]), we do not know if its statement becomes false upon removal of this hypothesis.

4.2. **Torsion modules.** We next provide criteria to determine whether a pro-discrete R-module is torsion.

Proposition 4.3. *A pro-discrete R-module $M = \varprojlim_n M_n$ is torsion in each of the following cases.*

(i) *Every $\mu_{R_n}(M_n)$ is finite and, for any sufficiently large n, the $\mathbb{Q}_p[\mathbb{Z}/(p^n)]$-module spanned by M_n has no non-zero free quotient.*
(ii) *The torsion-subgroup of each M_n has finite exponent, M_0 is a finitely generated abelian group and the R^p-module $\varprojlim_n(\mathbb{Z}_p \otimes_{\mathbb{Z}} M_n)$ is torsion.*

Proof. We first assume the hypotheses of (i). Then, for each n, the given isomorphism $M_{n-1} \cong (M_n)_{(n-1)}$ combines with a standard property of Fitting ideals to imply that $\varrho_{n,n-1}(\mathrm{Fitt}^0_{R_n}(M_n)) = \mathrm{Fitt}^0_{R_{n-1}}(M_{n-1})$. In particular, since $\mathrm{Fitt}^0_{R_n}(M_n)$ annihilates M_n, the limit $\varprojlim_n \mathrm{Fitt}^0_{R_n}(M_n)$ (with respect to the transition maps $\varrho_{n,n-1}$) is an ideal of R that annihilates M. To prove (i) it is therefore enough to show that this limit is non-zero, or equivalently that the ideal $\mathrm{Fitt}^0_{R_n}(M_n)$ is non-zero for some n (and hence for all sufficiently large n). In addition, since Fitting ideals commute with scalar extension, the ideal $\mathrm{Fitt}^0_{R_n}(M_n)$ vanishes if and only if $\mathbb{Q}_p \otimes_{\mathbb{Z}} \mathrm{Fitt}^0_{R_n}(M_n) = \mathrm{Fitt}^0_{\mathbb{Q}_p[\mathbb{Z}/(p^n)]}(\mathbb{Q}_p \otimes_{\mathbb{Z}} M_n)$ vanishes. Claim (i) is therefore true since $\mathrm{Fitt}^0_{\mathbb{Q}_p[\mathbb{Z}/(p^n)]}(\mathbb{Q}_p \otimes_{\mathbb{Z}} M_n)$ vanishes if and only if the $\mathbb{Q}_p[\mathbb{Z}/(p^n)]$-module $\mathbb{Q}_p \otimes_{\mathbb{Z}} M_n$ has no non-zero free quotient.

We now assume the conditions of (ii) and set $\mathbb{Z}' := \mathbb{Z}[1/p]$ and $M'_n := \mathbb{Z}' \otimes_{\mathbb{Z}} M_{n,\mathrm{tor}}$. Then, by passing to the limit over n of the obvious exact sequence $0 \to M'_n \to M_n \to \mathbb{Z}_p \otimes_{\mathbb{Z}} M_n$ we obtain an exact sequence of R-modules $0 \to M' \to M \to \widehat{M}$, with $M' := \varprojlim_n M'_n$ and $\widehat{M} := \varprojlim_n(\mathbb{Z}_p \otimes_{\mathbb{Z}} M_n)$ It is therefore enough to show that M' and $\widehat{M}$ are both torsion R-modules.

To deal with M' we write Δ_n for the subgroup of Γ_n of order p and $e_{(n)}$ for the idempotent $p^{-1}\sum_{h\in\Delta_n} h$ of $R_n[1/p]$. Then, upon taking Δ_n-coinvariance of the tautological short exact sequence $0 \to M_{n,\mathrm{tor}} \to M_n \to M_{n,\mathrm{tf}} \to 0$ one obtains an exact sequence

$$\mathrm{Tor}_1^{\mathbb{Z}[\Delta_n]}(\mathbb{Z}, M_{n,\mathrm{tf}}) \to (M_{n,\mathrm{tor}})_{(n-1)} \to M_{n-1} \to (M_{n,\mathrm{tf}})_{(n-1)} \to 0.$$

In particular, since $\mathrm{Tor}_1^{\mathbb{Z}[\Delta_n]}(\mathbb{Z}, M_{n,\mathrm{tf}}) \cong \hat{H}^{-2}(\Delta_n, M_{n,\mathrm{tf}})$ and the torsion subgroup $\hat{H}^{-1}(\Delta_n, M_{n,\mathrm{tf}})$ of $(M_{n,\mathrm{tf}})_{(n-1)}$ are both either trivial or of exponent p, the isomorphism $M_{n-1} \cong (M_n)_{(n-1)}$ restricts to give an isomorphism $M'_{n-1} \cong (M'_n)_{(n-1)}$. This isomorphism in turn induces a direct sum decomposition of R_n-modules

$$M'_n = (1 - e_{(n)})M'_n \oplus e_{(n)}M'_n \cong (1 - e_{(n)})M'_n \oplus M'_{n-1}$$

and one can use this to construct, by induction on n, a non-zero element of $R = \varprojlim_n R_n$ that annihilates M'. Explicitly, if x_{n-1} is a non-zero element of R_{n-1} that annihilates M'_{n-1} and $y_n \in R_n$ is any pre-image of x_{n-1} under $\varrho_{n,n-1}$, then the above decomposition implies $y_n e_{(n)}$ annihilates M'_n. We now fix an integer c that is divisible by p and such that $c \equiv 1 \pmod{t_n}$, where t_n is the exponent of M'_n. Then $x_n := y_n - cy_n(1 - e_{(n)})$ belongs to R_n, satisfies $\varrho_{n,n-1}(x_n) = x_{n-1}$ and annihilates M'_n since, for each $m \in M'_n$, one has

$$x_n(m) = y_n(1 - e_{(n)})(m) - cy_n(1 - e_{(n)})(m) = y_n(1 - e_{(n)})((1 - c)m) = y_n(1 - e_{(n)})(0) = 0.$$

To consider $\widehat{M}$, we note that the inverse system $(\mathbb{Z}_p \otimes_{\mathbb{Z}} M_n)_n$ induced by $(M_n)_n$ is pro-discrete, and hence that $\mathbb{Z}_p \otimes_{\mathbb{Z}} M_0$ identifies with $\mathbb{Z}_p \otimes_{R^p} \widehat{M}$. Since the given assumptions imply $\mathbb{Z}_p \otimes_{\mathbb{Z}} M_0$ is finitely-generated over $\mathbb{Z}_p$, we can therefore apply Nakayama's Lemma to the Noetherian ring R^p to deduce that $\widehat{M}$ is finitely generated. Since $\widehat{M}$ is also assumed to be a torsion R^p-module, we may therefore fix an element f of R^p that annihilates $\widehat{M}$. It is then enough to note that Proposition 3.4 implies $R \cap (R^p \cdot f) \neq (0)$. □

Remark 4.4. If $M = \varprojlim_n M_n$ is a finitely-generated torsion pro-discrete R-module, then the argument of Proposition 4.3(ii) implies $\widehat{M} := \varprojlim_n(\mathbb{Z}_p \otimes_{\mathbb{Z}} M_n)$ is a finitely generated torsion R^p-module. In particular, if we define the 'characteristic ideal' $\mathrm{char}_R(M)$ of M to be $R \cap \mathrm{char}_{R^p}(\widehat{M})$, then Proposition 3.4 implies this ideal determines $\mathrm{char}_{R^{\mathfrak{p}}}(\widehat{M})$. It is thus an interesting problem to understand if $\mathrm{char}_R(M)$ can be directly interpreted in terms of the structure of M as an R-module.

4.3. **Fitting ideals.** In this subsection, we clarify, under suitable hypotheses, the connection that exists between the initial Fitting ideal of an $R[H]$-module M and the corresponding Fitting ideals of the $R_n[H]$-modules $M_{(n)}$. Before stating the result, we recall that an $R[H]$-module M is said to be 'finitely ∞-presented' if, for every natural number t, there exists an exact sequence

$$M(t) \to M(t-1) \to \cdots \to M(0) \to M \to 0$$

of $R[H]$-modules in which each $M(i)$ is both finitely-generated and free. We also recall (from the beginning of the proof of Proposition 4.3) that for any finitely generated $R[H]$-module M one has $\varrho_{n,n-1}(\mathrm{Fit}^0_{R_n[H]}(M_{(n)})) = \mathrm{Fit}^0_{R_{n-1}[H]}(M_{(n-1)})$ and all $n \in \mathbb{N}$, and hence that both of the families

$$\big(\mathrm{Fit}^0_{R_n[H]}(M_{(n)}), \varrho_{n,n-1})_n \quad \text{and} \quad \big(\mathrm{Ann}_{R_n[H]}\big(\mathrm{Fit}^0_{R_n[H]}(M_{(n)})\big), \varrho_{n,n-1})_n$$

are inverse systems.

Proposition 4.5. *Fix a natural number d and an exact sequence of $R[H]$-modules*

$$P \to R[H]^d \to M \to 0 \tag{12}$$

in which P is the limit of a pro-discrete system $(P_n)_n$ in which each $R_n[H]$-module P_n is locally-free of rank d. Then the following claims are valid.

(i) *The $R[H]$-module M is finitely ∞-presented.*
(ii) *The $R[H]$-ideal $\mathrm{Fit}^0_{R[H]}(M)$ is pro-discrete and there exists a canonical short exact sequence*

$$0 \to \mathrm{Fit}^0_{R[H]}(M) \xrightarrow{\subseteq} \varprojlim\nolimits_n \mathrm{Fit}^0_{R_n[H]}(M_{(n)}) \xrightarrow{\xi_M} \varprojlim\nolimits^1_n \mathrm{Ann}_{R_n[H]}\big(\mathrm{Fit}^0_{R_n[H]}(M_{(n)})\big) \to 0 \tag{13}$$

in which both limits are taken with respect to the morphisms induced by the maps $\varrho_{n,n-1}$ and ξ_M is a natural connecting homomorphism.

Proof. We again set $\mathbb{A} := R[H]$ and $\mathbb{A}_n := R_n[H]$ for each n. If S denotes either $\mathbb{A}$ or $\mathbb{A}_n$, then for each finitely-presented S-module N we respectively write $\mathcal{F}(N)$ and $\mathcal{F}_n(N)$ for $\mathrm{Fit}^0_S(N)$.

We also note at the outset that the given hypotheses on P combine with Proposition 2.3(ii) to imply it is a finitely-generated pro-discrete $\mathbb{A}$-module. Then, Proposition 2.4(i) implies that, for every n, the $\mathbb{A}_n$-module $P_{(n)}$ identifies with P_n and so is locally-free of rank d.

Now, to prove (i), the result of [7, Prop. 5.1(ii)] reduces us to proving P is finitely-presented over $\mathbb{A}$ and hence, by Proposition 2.4(ii), to showing that $\mu_{\mathbb{A}_n}(\mathsf{Tor}^{\mathbb{A}}_1(\mathbb{A}_n, P))$ is bounded independently of n. We shall prove this by showing that each group $\mathsf{Tor}^{\mathbb{A}}_1(\mathbb{A}_n, P)$ vanishes. To do this we fix n and note that, for each integer $t > 1$, Proposition 2.3(ii) implies the existence of a free $\mathbb{A}$-submodule $P' = P'(n,t)$ of P that has rank d and is such that the natural map $P'_{(n)} \xrightarrow{\nu_n} P_{(n)}$ has finite cokernel of order prime-to-pt and is therefore also injective (since $P'_{(n)}$ and $P_{(n)}$ are both locally-free of rank d). Then the tautological exact sequence $0 \to P' \to P \to P/P' \to 0$ gives rise to an exact sequence

$$\mathsf{Tor}^{\mathbb{A}}_1(\mathbb{A}_n, P') \to \mathsf{Tor}^{\mathbb{A}}_1(\mathbb{A}_n, P) \xrightarrow{\nu'_n} \mathsf{Tor}^{\mathbb{A}}_1(\mathbb{A}_n, P/P') \to P'_{(n)} \xrightarrow{\nu_n} P_{(n)}.$$

Now, since the first term in this sequence vanishes (as P' is free) and ν_n is injective, the map ν'_n is bijective. In addition, the $\mathbb{A}_n$-module $\mathsf{Tor}^{\mathbb{A}}_1(\mathbb{A}_n, P/P')$ is annihilated by the image of $\mathcal{F}(P/P')$ under the projection map $\varrho_n : \mathbb{A} \to \mathbb{A}_n$. Thus, since $\varrho_n\big(\mathcal{F}(P/P')\big) = \mathcal{F}_n((P/P')_{(n)})$ (by standard functorial properties of Fitting ideals) and $(P/P')_{(n)}$ is isomorphic to the finite module $\mathrm{cok}(\nu_n)$, it follows that $\mathsf{Tor}^{\mathbb{A}}_1(\mathbb{A}_n, P/P')$, and hence also $\mathsf{Tor}^{\mathbb{A}}_1(\mathbb{A}_n, P)$, is annihilated by a natural number that is prime to t. Since t can be chosen arbitrarily, this in turn implies $\mathsf{Tor}^{\mathbb{A}}_1(\mathbb{A}_n, P)$ vanishes, as required to prove (i).

To prove (ii) we note that, since M is finitely-presented, the $\mathbb{A}$-ideal $\mathcal{F}(M)$ is automatically finitely-generated and hence also, by Proposition 2.4(iii), pro-discrete. To compute $\mathcal{F}(M)$, we are therefore reduced to considering its quotients $\mathcal{F}(M)_{(n)}$ for $n \in \mathbb{N}$. To do this, we fix n and, for each integer $t > 1$, use the free $\mathbb{A}$-submodule $P' = P'(n,t)$ of P constructed above. We also write θ for the map $P \to \mathbb{A}^d$ in (12) and M' for the cokernel of the composite map $P' \subseteq P \xrightarrow{\theta} \mathbb{A}^d$. We then claim that there is an equality

$$\mathsf{Tor}_1^{\mathbb{A}}(\mathbb{A}_n, \mathbb{A}/\mathcal{F}(M')) = \mathrm{Ann}_{\mathbb{A}_n}(\mathcal{F}_n(M'_{(n)})). \tag{14}$$

To verify this we note that, as P' is free of rank d, $\mathcal{F}(M')$ is a principal $\mathbb{A}$-ideal and so, after choosing a generator λ, there exists an exact sequence of $\mathbb{A}$-modules $0 \to \mathbb{A} \xrightarrow{\times\lambda} \mathbb{A} \to \mathbb{A}/\mathcal{F}(M') \to 0$. This exact sequence induces an identification

$$\mathsf{Tor}_1^{\mathbb{A}}(\mathbb{A}_n, \mathbb{A}/\mathcal{F}(M')) = \ker\big(\mathbb{A}_n \xrightarrow{\varrho_n(\lambda)} \mathbb{A}_n\big) = \{r \in \mathbb{A}_n : r \cdot \varrho_n(\lambda) = 0\} = \mathrm{Ann}_{\mathbb{A}_n}(\varrho_n(\mathcal{F}(M'))),$$

where the last equality is valid since $\varrho_n(\mathcal{F}(M')) = \mathbb{A}_n \cdot \varrho_n(\lambda)$. To deduce (14), it is then enough to note that $\varrho_n(\mathcal{F}(M')) = \mathcal{F}_n(M'_{(n)})$.

We note next that $\mathcal{F}(M') \subseteq \mathcal{F}(M)$ and hence that there exists a tautological short exact sequence $0 \to \mathcal{F}(M)/\mathcal{F}(M') \to \mathbb{A}/\mathcal{F}(M') \to \mathbb{A}/\mathcal{F}(M) \to 0$. Taken in conjunction with (14), this sequence gives rise to an exact sequence of $\mathbb{A}_n$-modules

$$\mathsf{Tor}_1^{\mathbb{A}}(\mathbb{A}_n, \mathcal{F}(M)/\mathcal{F}(M')) \xrightarrow{\alpha'} \mathrm{Ann}_{\mathbb{A}_n}(\mathcal{F}_n(M'_{(n)})) \xrightarrow{\alpha} \mathsf{Tor}_1^{\mathbb{A}}(\mathbb{A}_n, \mathbb{A}/\mathcal{F}(M)) \to \big(\mathcal{F}(M)/\mathcal{F}(M')\big)_{(n)}. \tag{15}$$

Now, since P/P' is the kernel of the surjective map $M' \to M$, a standard multiplication property of Fitting ideals implies that the quotient $\mathcal{F}(M)/\mathcal{F}(M')$ is annihilated by the ideal $\mathcal{F}(P/P')$. The two end terms in the above sequence are therefore annihilated by the ideal $\varrho_n(\mathcal{F}(P/P')) = \mathcal{F}_n((P/P')_{(n)})$ of $\mathbb{A}_n$. In particular, since $(P/P')_{(n)}$ is finite and of order prime-to-t, we can deduce that the index c of $\mathrm{Ann}_{\mathbb{A}_n}(\mathcal{F}_n(M'_{(n)}))$ in $\mathrm{Ann}_{\mathbb{A}_n}(\mathcal{F}_n(M_{(n)}))$ is finite and prime-to-t, and also that both end terms in the exact sequence (15) are annihilated by a natural number c' that is prime-to-t. In particular, since $\mathrm{Ann}_{\mathbb{A}_n}(\mathcal{F}_n(M'_{(n)}))$ is $\mathbb{Z}$-free, its submodule $\ker(\alpha) = \mathrm{im}(\alpha')$ must therefore vanish and so we can identify $\mathrm{Ann}_{\mathbb{A}_n}(\mathcal{F}_n(M'_{(n)}))$ with its image under α.

In addition, if we now repeat this argument after replacing t by the product cc', we can deduce the existence of a $\mathbb{A}$-module M'' with the property that

$$\mathrm{Ann}_{\mathbb{A}_n}(\mathcal{F}_n(M_{(n)})) = \mathrm{Ann}_{\mathbb{A}_n}(\mathcal{F}_n(M'_{(n)})) + \mathrm{Ann}_{\mathbb{A}_n}(\mathcal{F}_n(M''_{(n)})) = \mathsf{Tor}_1^{\mathbb{A}}(\mathbb{A}_n, \mathbb{A}/\mathcal{F}(M)).$$

Taking account of this equality, the tautological short exact sequence

$$0 \to \mathcal{F}(M) \to \mathbb{A} \to (\mathbb{A}/\mathcal{F}(M)) \to 0$$

combines with the identification $\varrho_n(\mathcal{F}(M)) = \mathcal{F}_n(M_{(n)})$ to induce, for every n, a canonical exact commutative diagram of $\mathbb{A}_n$-modules

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ann}_{\mathbb{A}_n}(\mathcal{F}_n(M_{(n)})) & \longrightarrow & \mathcal{F}(M)_{(n)} & \longrightarrow & \mathcal{F}_n(M_{(n)}) & \longrightarrow & 0 \\
 & & {\scriptstyle \varrho_{n,n-1}}\big\downarrow & & \big\downarrow & & {\scriptstyle \varrho_{n,n-1}}\big\downarrow & & \\
0 & \longrightarrow & \mathrm{Ann}_{\mathbb{A}_n}(\mathcal{F}_{n-1}(M_{(n-1)})) & \longrightarrow & \mathcal{F}(M)_{(n-1)} & \longrightarrow & \mathcal{F}_{n-1}(M_{(n-1)}) & \longrightarrow & 0
\end{array}$$

in which the central vertical map is the canonical projection. Upon passing to the limit over n of these sequences, and recalling that $\mathcal{F}(M)$ is pro-discrete, we therefore obtain an exact sequence of the form (13), as required to complete the proof. □

Remark 4.6. If N is any pro-discrete ideal of $R[H]$ for which every $R_n[H]$-module $N_{(n)}$ is locally-free of rank 1, then N is finitely-generated and one has a canonical exact sequence of $R[H]$-modules

$$0 \to N \xrightarrow{\subseteq} \varprojlim_n \varrho_n(N) \xrightarrow{\xi_N} \varprojlim_n^1 \mathrm{Ann}_{R_n[H]}\big(\varrho_n(N)\big) \to 0.$$

To see this, note that $N = \varprojlim_n N_n$ for a pro-discrete family $(N_n)_n$ of $\mathbb{A}_n$-modules. In particular, for every n the surjectivity of the canonical map $N_{(n)} \to N_n$ combines with the Swan-Forster Theorem (via [7, Rem 3.14]) and the fact $N_{(n)}$ is locally-free of rank 1 to imply that $\mu_{\mathbb{A}_n}(N_n) \leq 2$. The ideal N is thus finitely-generated by Proposition 2.3(ii) and so, for each n, the map $N_{(n)} \to N_n$ is bijective by Proposition 2.4(i). It follows that the module $M = \mathbb{A}/N$ has a presentation of the form (12) with $P = N$ and $d = 1$. In addition, $\mathcal{F}(\mathbb{A}/N) = N$ and $\mathcal{F}_n((\mathbb{A}/N)_{(n)}) = \mathcal{F}_n(\mathbb{A}_n/\varrho_n(N)) = \varrho_n(N)$ and so the claimed exact sequence follows as a special case of (13).

The derived limit in the exact sequence of Proposition 4.5 is usually non-zero. For example, if H is trivial and, for any non-negative integer n, one sets $N := R \cdot \varpi_n$, then the short exact sequence in Remark 4.6 recovers the fundamental exact sequence of [7, Prop. 2.1(ii)]. The following more general observation about derived limits will be useful in the sequel.

Lemma 4.7. *Let $(Y_n)_{n\in\mathbb{N}}$ be an inverse system of finitely-generated abelian groups. Then $\varprojlim_n^1 Y_n$ is divisible and its torsion subgroup is supported on primes ℓ for which $\varprojlim_n(\mathbb{Z}_\ell \otimes_{\mathbb{Z}} Y_{n,\mathrm{tf}})$ is non-zero.*

Proof. Since each group $Y_{n,\mathrm{tor}}$ is finite, the Mittag-Leffler criterion implies that $\varprojlim_n^i Y_{n,\mathrm{tor}}$ vanishes for both $i = 1$ and $i = 2$. By passing to the limit over n of the tautological short exact sequences $0 \to Y_{n,\mathrm{tor}} \to Y_n \to Y_{n,\mathrm{tf}} \to 0$, one deduces that the natural map $\varprojlim_n^1 Y_n \to \varprojlim_n^1 Y_{n,\mathrm{tf}}$ is bijective. Then, for each prime ℓ there is a short exact sequence

$$0 \to Y_{n,\mathrm{tf}} \to \mathbb{Z}_\ell \otimes_{\mathbb{Z}} Y_{n,\mathrm{tf}} \to (\mathbb{Z}_\ell/\mathbb{Z}) \otimes_{\mathbb{Z}} Y_{n,\mathrm{tf}} \to 0.$$

These sequences are compatible with the natural transition maps as n varies. Hence, since each module $\mathbb{Z}_\ell \otimes_{\mathbb{Z}} Y_{n,\mathrm{tf}}$ is compact (and the relevant transition morphisms are continuous), upon passing to the inverse limit one obtains an exact sequence

$$\varprojlim_n\big(\mathbb{Z}_\ell \otimes_{\mathbb{Z}} Y_{n,\mathrm{tf}}\big) \to \varprojlim_n\big((\mathbb{Z}_\ell/\mathbb{Z}) \otimes_{\mathbb{Z}} Y_{n,\mathrm{tf}}\big) \to \varprojlim_n^1 Y_n \to 0.$$

Now, since $\mathbb{Z}_\ell/\mathbb{Z}$ is uniquely ℓ-divisible, the second term in this sequence is uniquely ℓ-divisible. The exact sequence therefore implies that the group $\varprojlim_n^1 Y_n$ is ℓ-divisible, respectively uniquely ℓ-divisible if $\varprojlim_n\big(\mathbb{Z}_\ell \otimes_{\mathbb{Z}} Y_{n,\mathrm{tf}}\big)$ vanishes, as claimed. ◻

Remark 4.8. In view of Lemma 4.7, the short exact sequence (13) (and hence also that displayed in Remark 4.6) raises the question of whether finitely generated ideals of $R[H]$ are uniquely determined by their projections to $R_n[H]$ for each n. However, the chain of ideals constructed in Proposition 3.2(iii) shows that this is in general far from true.

4.4. **Derived limits of complexes.** In this subsection, we review the construction of derived limits of complexes (as presented, for example, in [30]) in the setting of pro-discrete modules. The results obtained here will then be used in our study of Weil-étale cohomology in §6.

For a commutative ring B we write $\mathsf{D}(B)$ for the derived category of B-modules and, if B is Noetherian, we further write $\mathsf{D}^{\mathrm{perf}}(B)$ for the full triangulated subcategory of $\mathsf{D}(B)$ comprising complexes that are isomorphic to a bounded complex of finitely generated projective B-modules.

For convenience, we shall also again set

$$\mathbb{A}_n := R_n[H] \;\text{ for }\; n \in \mathbb{N}_0 \;\text{ and }\; \mathbb{A} := R[H] = \varprojlim_{\varrho_{n,n-1}} \mathbb{A}_n.$$

We then assume to be given a family of complexes $(X_n^\bullet)_{n\in\mathbb{N}_0}$ of $\mathbb{A}$-modules satisfying the following hypothesis.

Hypothesis 4.9.

(i) *For $n \in \mathbb{N}_0$, the action of $\mathbb{A}$ on $X_n^\bullet$ factors through $\mathbb{A}_n$.*

(ii) *For $n \in \mathbb{N}$, there exists an isomorphism $\theta_n : \mathbb{A}_{n-1} \otimes^{\mathsf{L}}_{\mathbb{A}_n} X_n^\bullet \cong X_{n-1}^\bullet$ in $\mathsf{D}(\mathbb{A}_{n-1})$.*

In the context of (ii), we will also write θ_n for the induced composite morphism in $\mathsf{D}(\mathbb{A}_n)$

$$X_n^\bullet \to \mathbb{A}_{n-1} \otimes^{\mathsf{L}}_{\mathbb{A}_n} X_n^\bullet \cong X_{n-1}^\bullet.$$

Definition 4.10. The 'derived limit' of a family $(X_n^\bullet)_{n\in\mathbb{N}}$ of complexes satisfying Hypothesis 4.9 is a complex $\mathsf{R}\varprojlim_n X_n^\bullet$ that lies in an exact triangle in $\mathsf{D}(\mathbb{A})$ of the form

$$\mathsf{R}\varprojlim_n X_n^\bullet \to \prod_{n\in\mathbb{N}} X_n^\bullet \xrightarrow{(1-\theta_n)_n} \prod_{n\in\mathbb{N}} X_n^\bullet \to \cdot \tag{16}$$

(cf. [30, Def. 13.34.1]). We recall that such a complex always exists and is unique up to isomorphism in $\mathsf{D}(\mathbb{A})$ (cf. [30, Lem. 15.88.3, Lem. 15.88.5 and Rem. 15.88.6]).

The following result describes some properties of this construction that will be useful for us.

Proposition 4.11. *Assume there exist integers a and b with $a \le b$ such that, for every $n \in \mathbb{N}_0$, the group $H^i(X_n^\bullet)$ vanishes for both $i < a$ and $i > b$. Then the following claims are valid.*

(i) $H^i(\mathsf{R}\varprojlim_n X_n^\bullet)$ vanishes for both $i < a$ and $i > b$. Further, there exists a canonical isomorphism $H^a(\mathsf{R}\varprojlim_n X_n^\bullet) \cong \varprojlim_{n\in\mathbb{N}} H^a(X_n^\bullet)$ and, for $i > a$, a canonical short exact sequence

$$0 \to \varprojlim^1_n H^{i-1}(X_n^\bullet) \to H^i(\mathsf{R}\varprojlim_n X_n^\bullet) \to \varprojlim_n H^i(X_n^\bullet) \to 0,$$

where the respective limits are taken with respect to the morphisms $H^{i-1}(\theta_n)$ and $H^i(\theta_n)$.

(ii) Suppose to be given two further families $(Y_n^\bullet, \theta'_n)$ and $(Z_n^\bullet, \theta''_n)$ satisfying Hypothesis 4.9, together with a collection of compatible exact triangles $X_n^\bullet \to Y_n^\bullet \to Z_n^\bullet \to \cdot$ in $\mathsf{D}(\mathbb{A}_n)$. Then there is an exact triangle $\mathsf{R}\varprojlim_n X_n^\bullet \to \mathsf{R}\varprojlim_n Y_n^\bullet \to \mathsf{R}\varprojlim_n Z_n^\bullet \to \cdot$ in $\mathsf{D}(\mathbb{A})$.

(iii) For each m there exists a canonical morphism $\mathbb{A}_m \otimes^{\mathsf{L}}_{\mathbb{A}} \mathsf{R}\varprojlim_n X_n^\bullet \to X_m^\bullet$ in $\mathsf{D}(\mathbb{A}_m)$.

(iv) Assume that $b = a + 1$ and that all of the following conditions are satisfied:

(a) *for all n, the complex $X_n^\bullet$ belongs to $\mathsf{D}^{\mathrm{perf}}(\mathbb{A}_n)$;*

(b) *for all n, the module $H^a(X_n^\bullet)$ is $\mathbb{Z}$-free;*

(c) *for all n, the $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{A}_n$-modules spanned by $H^a(X_n^\bullet)$ and $H^{a+1}(X_n^\bullet)$ are isomorphic.*

(d) *$\varprojlim_n H^{a+1}(X_n^\bullet)$ is a finitely-generated $\mathbb{A}$-module.*

Then $\mathsf{R}\varprojlim_n X_n^\bullet$ is isomorphic in $\mathsf{D}(\mathbb{A})$ to a complex $P \to \Pi$ of $\mathbb{A}$-modules in which Π is free of finite rank and occurs in degree $a + 1$ and P is both finitely-generated and such that each $\mathbb{A}_n$-module $P_{(n)}$ is locally-free of the same rank as Π. If p is not exceptional, then P is also projective. Moreover, in all cases the morphisms in (iii) are isomorphisms and the induced maps of $\mathbb{A}_m$-modules $H^{a+1}(\mathsf{R}\varprojlim_n X_n^\bullet)_{(m)} \to H^{a+1}(X_m^\bullet)$ are bijective.

Proof. The long exact sequence of cohomology of the exact triangle (16) gives in each degree k a short exact sequence

$$0 \to \varprojlim^1_n H^{k-1}(X_n^\bullet) \to H^k(\mathsf{R}\varprojlim_n X_n^\bullet) \to \varprojlim_n H^k(X_n^\bullet) \to 0, \tag{17}$$

in which each limit is taken with respect to the transition morphisms induced by the respective maps $H^i(\theta_n)$. The isomorphism in the second assertion of (i), together with the acyclicity of $\mathsf{R}\varprojlim_n X_n^\bullet$ in all degrees less than a and greater than $b + 1$, follow directly from these sequences. The acyclicity in degree $b + 1$ follows in the same way since each morphism $H^b(\theta_n) : H^b(X_n^\bullet) \to H^b(X_{n-1}^\bullet)$ is surjective (as $X_n^\bullet$ is acyclic in degrees greater than b).

To prove (ii) we note that restriction of scalars along $\mathbb{A} \to \mathbb{A}_n$ is an exact functor between the respective module categories and hence that each of the given triangles is exact in $\mathsf{D}(\mathbb{A})$. In particular, since the category of $\mathbb{A}$-modules has exact products, the given assumptions imply the existence of a commutative diagram of exact triangles in $\mathsf{D}(\mathbb{A})$ of the form

$$\begin{tikzcd}
 & \cdot & \cdot & \\
\mathsf{R}\varprojlim_n Z_n^\bullet \arrow[r] & \prod_{n\in\mathbb{N}} Z_n^\bullet \arrow[r, "(1-\theta''_n)_n"] \arrow[u] & \prod_{n\in\mathbb{N}} Z_n^\bullet \arrow[r] \arrow[u] & \cdot \\
\mathsf{R}\varprojlim_n Y_n^\bullet \arrow[r] & \prod_{n\in\mathbb{N}} Y_n^\bullet \arrow[r, "(1-\theta'_n)_n"] \arrow[u] & \prod_{n\in\mathbb{N}} Y_n^\bullet \arrow[r] \arrow[u] & \cdot \\
\mathsf{R}\varprojlim_n X_n^\bullet \arrow[r] & \prod_{n\in\mathbb{N}} X_n^\bullet \arrow[r, "(1-\theta_n)_n"] \arrow[u] & \prod_{n\in\mathbb{N}} X_n^\bullet \arrow[r] \arrow[u] & \cdot
\end{tikzcd}$$

in which each row is the corresponding case of (16). By applying the Octahedral axiom to this diagram, one obtains an exact triangle as in (ii).

For each $m > n$ we write $\theta_{m,n} : X_m^\bullet \to X_n^\bullet$ for the composite morphism $\theta_{n+1} \circ \cdots \circ \theta_{m-1} \circ \theta_m$. Then, to prove (iii), we use the commutative square in $\mathsf{D}(\mathbb{A})$

$$\begin{tikzcd}
\mathbb{A}_n \otimes^{\mathsf{L}}_{\mathbb{A}} \prod_{m\in\mathbb{N}} X_m^\bullet \arrow[r, "(1-\theta_m)_m"] \arrow[d, "(\theta_{m,n})_{m>n}"] & \mathbb{A}_n \otimes^{\mathsf{L}}_{\mathbb{A}} \prod_{m\in\mathbb{N}} X_m^\bullet \arrow[d, "(\theta_{m,n})_{m>n}"] \\
\prod_{m>n} X_n^\bullet \arrow[r, "(1-\mathrm{id}_m)_m"] & \prod_{m>n} X_n^\bullet
\end{tikzcd}$$

in which id_m denotes the identity morphism from $X_n^\bullet$ in the m-th component of the direct product to $X_n^\bullet$ in the $(m-1)$-st component of the product. This diagram combines with the exact triangle (16) to induce a morphism from $\mathbb{A}_n \otimes^{\mathsf{L}}_{\mathbb{A}} \mathsf{R}\varprojlim_m X_m^\bullet$ to the derived limit of the constant system $(X_n^\bullet, \mathrm{id})_{m\in\mathbb{N}_0}$. This in turn induces a morphism of the required sort since the latter system is $\varprojlim_n$-acyclic and so [30, Lem. 15.88.1(5) and Lem. 15.88.3] combine to imply its derived limit is represented by $X_n^\bullet$.

To prove (iv) we claim first that the R-module $\varprojlim_n H^{a+1}(X_n^\bullet)$ is pro-discrete. Indeed, this is true since the given isomorphism θ_n combines with the fact that $X_n^\bullet$ and $X_{n-1}^\bullet$ are each acyclic in degrees greater than $a+1$ to induce an isomorphism of R_{n-1}-modules

$$H^{a+1}(X_n^\bullet)_{(n-1)} \cong H^{a+1}(X_{n-1}^\bullet). \tag{18}$$

We next note that condition (d) allows us to fix a free $\mathbb{A}$-module Π of finite rank t, together with a surjective homomorphism

$$\pi : \Pi \to \varprojlim_n H^{a+1}(X_n^\bullet).$$

Then, for each $n \in \mathbb{N}_0$ we write Π_n for the free, rank t, $\mathbb{A}_n$-module $\mathbb{A}_n \otimes_{\mathbb{A}} \Pi$ and we note that, as $\varprojlim_n H^{a+1}(X_n^\bullet)$ is pro-discrete, the map $\pi_n : \Pi_n \to H^{a+1}(X_n^\bullet)$ induced by π is surjective (cf. [7, Rem. 3.3]). Now, by a standard argument (as in [8, §5.4]), the conditions (a), (b) and (c) combine to imply $X_n^\bullet$ is isomorphic in $\mathsf{D}(\mathbb{A}_n)$ to a complex $\Pi_n^\bullet$ of the form $P_n \xrightarrow{\phi_n} \Pi_n$, where P_n is a finitely-generated locally-free (and hence projective) $\mathbb{A}_n$-module of rank t placed in degree a, and $\mathrm{cok}(\phi_n)$ is identified with $H^{a+1}(X_n^\bullet)$ via π_n. The composite morphism

$$\Pi_n^\bullet \cong X_n^\bullet \xrightarrow{\theta_n} X_{n-1}^\bullet \cong \Pi_{n-1}^\bullet$$

in $\mathsf{D}(\mathbb{A}_n)$ can then be represented by a concrete morphism of complexes $\theta_n^\bullet = (\theta_n^a, \theta_n^{a+1})$ and, by changing $\theta_n^\bullet$ by a homotopy (if necessary), we can assume that θ_n^{a+1} is the natural projection map $\rho_n : \Pi_n \to \Pi_{n-1}$. We now consider the following commutative diagram of $\mathbb{A}_{n-1}$-modules

$$\begin{tikzcd}
(P_n)_{(n-1)} \arrow[r, "(\theta_n^a)_{(n-1)}"] \arrow[d, equal] & P_{n-1} \arrow[d, "x\mapsto(x,0)"] & \\
(P_n)_{(n-1)} \arrow[r, "((\theta_n^a)_{(n-1)},(\phi_n)_{(n-1)})"] & P_{n-1} \oplus (\Pi_n)_{(n-1)} \arrow[r, "(\phi_{n-1},-(\theta_n^{a+1})_{(n-1)})"] \arrow[d, "(y,z)\mapsto -z"'] & \Pi_{n-1} \arrow[d, equal] \\
& (\Pi_n)_{(n-1)} \arrow[r, "(\theta_n^{a+1})_{(n-1)}"] & \Pi_{n-1}.
\end{tikzcd}$$

The central row of this diagram is the mapping cone of the morphism $\mathbb{A}_{n-1} \otimes_{\mathbb{A}_n} \Pi_n^\bullet \to \Pi_{n-1}^\bullet$ induced by $\theta_n^\bullet$. In particular, since $\theta_n^\bullet$ represents θ_n, this mapping cone is acyclic and so the central row is exact. Since the morphism $(\theta_n^{a+1})_{(n-1)}$ on the bottom row is induced by ρ_n and so is bijective, and the central column is a short exact sequence, a diagram chase shows that the morphism $(\theta_n^a)_{(n-1)}$ in the upper row is also bijective. It follows that the family $(P_n, \theta_n^a)_n$ is a pro-discrete system and so Propositions 2.3(ii) and 4.1 combine to imply that its limit $P := \varprojlim_n P_n$ is a finitely-generated, pro-discrete $\mathbb{A}$-module that is also, unless p is exceptional, projective. From Proposition 2.4(i) we also know that the canonical map $P_{(n)} \to P_n$ is bijective for every n.

Writing ϕ_∞ for the limit homomorphism $\varprojlim_n \phi_n$, we next claim that the complex $\Pi^\bullet$ given by $P \xrightarrow{\phi_\infty} \Pi$, where P is placed in degree a, is isomorphic to $\mathsf{R}\varprojlim_n X_n^\bullet$ in $\mathsf{D}(\mathbb{A})$.

To show this we note that, since $\theta_n^\bullet$ represents θ_n in $\mathsf{D}(\mathbb{A}_n)$, the derived limit $\mathsf{R}\varprojlim_n \Pi_n^\bullet$ of the family $(\Pi_n^\bullet, \theta_n^\bullet)$ is isomorphic in $\mathsf{D}(\mathbb{A})$ to $\mathsf{R}\varprojlim_n X_n^\bullet$. It is therefore enough to show that $\Pi^\bullet$ is isomorphic in $\mathsf{D}(\mathbb{A})$ to $\mathsf{R}\varprojlim_n \Pi_n^\bullet$.

Write $\mathsf{Mod}^{\mathbb{N}}(\mathbb{A})$ for the abelian category of inverse systems (M_n, ψ_n) of $\mathbb{A}$-modules where each M_n is naturally a $\mathbb{A}_n$-module and $\psi_n : M_n \to M_{n-1}$ is a homomorphism of $\mathbb{A}$-modules. Then, since $\mathsf{Mod}^{\mathbb{N}}(\mathbb{A})$ has enough injectives, the natural inverse limit functor $\varprojlim_n : \mathsf{Mod}^{\mathbb{N}}(\mathbb{A}) \to \mathsf{Mod}(\mathbb{A})$ induces a functor $\mathsf{R}\varprojlim'_n : \mathsf{D}(\mathsf{Mod}^{\mathbb{N}}(\mathbb{A})) \to \mathsf{D}(\mathbb{A})$ on the corresponding derived categories. The key point now is that the family $(\Pi_n^\bullet, \theta_n^\bullet)$ defines an object of $\mathsf{D}(\mathsf{Mod}^{\mathbb{N}}(\mathbb{A}))$ with the property that both occurring systems (P_n, θ_n^a) and (Π_n, θ_n^{a+1}) are $\varprojlim_n$-acyclic since every transition map θ_n^a and θ_n^{a+1} is surjective. It follows from [30, Lem. 15.88.1(5)] that $\mathsf{R}\varprojlim'_n \Pi_n^\bullet$ is represented by the complex $\Pi^\bullet$ and hence, by [30, Lem. 15.88.3], that $\mathsf{R}\varprojlim_n \Pi_n^\bullet$ is naturally isomorphic to $\Pi^\bullet$ in $\mathsf{D}(\mathbb{A})$, as required. This completes the proof of all but the final assertion of (iv). The final assertion of (iv) is then true since, in this case, the morphism in (iii) coincides with the composite morphism

$$\mathsf{R}\varprojlim_n X_n^\bullet \cong \Pi^\bullet \xrightarrow{x\mapsto 1\otimes x} \mathbb{A}_m \otimes_{\mathbb{A}} \Pi^\bullet = \Pi_m^\bullet \cong X_m^\bullet,$$

and so the induced map of $\mathbb{A}_m$-modules $H^{a+1}(\mathsf{R}\varprojlim_n X_n^\bullet)_{(m)} \to H^{a+1}(X_m^\bullet)$ is bijective as $\mathrm{cok}(\phi_\infty)_{(m)}$ identifies with $\mathrm{cok}(\phi_m)$. □

5. Divisor class groups

For any global function field E we write C_E for the unique geometrically irreducible smooth projective curve that has function field E, and E^{con} for the constant $\mathbb{Z}_p$-extension of E.

In this section we fix such a field k of characteristic p together with a finite, possibly empty, set of places Σ of k. For a finite extension F of k, we write Σ_F for the set of places of F that lie above those in Σ. If $\Sigma \neq \emptyset$, we write $\mathcal{O}_{F,\Sigma}$ for the subring of F comprising elements integral at all places outside Σ_F, $B_\Sigma(F)$ for $\bigoplus_{w\in\Sigma_F}\mathbb{Z}$ and $B^0_\Sigma(F)$ for the kernel of the map $B_\Sigma(F)\to\mathbb{Z}$ sending each $(n_w)_w$ to $\sum_w n_w$. We define a finite abelian group

$$A_\Sigma(F) := \begin{cases}\mathrm{Pic}(\mathcal{O}_{F,\Sigma}), & \text{if } \Sigma\neq\emptyset\\ \mathrm{Pic}^0(C_F), & \text{if } \Sigma=\emptyset.\end{cases}$$

For each prime ℓ, we set $A^\ell_\Sigma(F) := A_\Sigma(F)[\ell^\infty]$. We then set $A'_\Sigma(F) := \bigoplus_{\ell\neq p} A^\ell_\Sigma(F)$, so that $A_\Sigma(F) = A^p_\Sigma(F)\oplus A'_\Sigma(F)$.

We now fix a $\mathbb{Z}_p$-extension $\mathcal{K}$ of k that is ramified at only finitely many places. For $n\in\mathbb{N}_0$, we write $\mathcal{K}_n$ for the extension of k of degree p^n inside $\mathcal{K}$. Then, for $n\in\mathbb{N}$, the field-theoretic norm map $\mathrm{N}_n:\mathcal{K}_n^\times\to\mathcal{K}_{n-1}^\times$ induces a map $A_\Sigma(\mathcal{K}_n)\to A_\Sigma(\mathcal{K}_{n-1})$ (that we also denote by N_n) and we set

$$A_\Sigma(\mathcal{K}) := \varprojlim\nolimits_{\mathrm{N}_n} A_\Sigma(\mathcal{K}_n) \quad\text{and}\quad A^\ell_\Sigma(\mathcal{K}) := \varprojlim\nolimits_{\mathrm{N}_n} A^\ell_\Sigma(\mathcal{K}_n) \ \text{ for each } \ell.$$

We fix a topological generator of $G_{\mathcal{K}/k} := \mathrm{Gal}(\mathcal{K}/k)$, thereby regarding $A_\Sigma(\mathcal{K})$ and each $A^\ell_\Sigma(\mathcal{K})$ as an R-module, and our aim is then to use the theory of pro-discrete R-modules to study $A_\Sigma(\mathcal{K})$.

To state the result of this section, we use the notion of a finitely ∞-presented module, as recalled just before Proposition 4.5. In addition, for each prime $\ell\neq p$ we define the 'rank' $\mathrm{r}(X)$ of a finite abelian group X of ℓ-power order via the equality $\ell^{\mathrm{r}(X)} = |X/\ell|$ and write $t_n(\ell)$ for the number of places of the field $\mathbb{Q}(e^{2\pi i/p^n})$ that lie above ℓ.

Theorem 5.1. *Fix a $\mathbb{Z}_p$-extension $\mathcal{K}/k$ and a finite (possibly empty) set of places Σ of k.*

(i) *$A_\Sigma(\mathcal{K})$ is pro-discrete and torsion as an R-module.*
(ii) *$\mu_R\big(A^p_\Sigma(\mathcal{K})\big)$ is finite if and only if $A^p_\Sigma(\mathcal{K})^{\omega_n=0}$ is finite for all $n\in\mathbb{N}_0$.*
(iii) *$\mu_R\big(A^\ell_\Sigma(\mathcal{K})\big)$ is finite for $\ell\neq p$ if and only if $\mathrm{r}(A^\ell_\Sigma(\mathcal{K}_n)) = O(p^n)$ as $n\to\infty$.*
(iv) *$A_\Sigma(\mathcal{K})$ is finitely ∞-presented as an R-module if and only if $\mu_R\big(A_\Sigma(\mathcal{K})\big)$ is finite.*
(v) *$A_\Sigma(\mathcal{K})$ is a pro-discrete, torsion and finitely ∞-presented R-module if $A^p_\Sigma(\mathcal{K})^{\omega_n=0}$ is finite for every n and also $t_n(\ell)\mathrm{r}(A^\ell_\Sigma(\mathcal{K}_n))/p^n$ is bounded independently of both $\ell\in\mathcal{P}\setminus\{p\}$ and n.*
(vi) *The conditions in (v) are satisfied if $\mathcal{K} = k^{\mathrm{con}}$.*

Proof. Since $A^\ell_\Sigma(\mathcal{K})$ is the maximal pro-ℓ-subgroup of $A_\Sigma(\mathcal{K})$, one has $A_\Sigma(\mathcal{K}) = A^p_\Sigma(\mathcal{K})\oplus A'_\Sigma(\mathcal{K})$, with $A'_\Sigma(\mathcal{K}) := \varprojlim_{\mathrm{N}_n} A'_\Sigma(\mathcal{K}_n) = \prod_{\ell\neq p} A^\ell_\Sigma(\mathcal{K})$. To prove $A_\Sigma(\mathcal{K})$ is pro-discrete, it is therefore enough to prove that the modules $A^p_\Sigma(\mathcal{K})$ and $A'_\Sigma(\mathcal{K})$ are each pro-discrete.

To analyse $A^p_\Sigma(\mathcal{K})$, we use the following notation for each $n\in\mathbb{N}_0$: we write $\delta^\Sigma_n : B_\Sigma(\mathcal{K}_n)\to\mathbb{Z}$ is the homomorphism sending each $w\in\Sigma_{\mathcal{K}_n}$ to $\deg_k(w)$, m^Σ_n for the natural number that generates $\mathrm{im}(\delta^\Sigma_n)$; Δ^Σ_n for the divisor map $\mathcal{O}^\times_{\mathcal{K}_n,\Sigma}\to\ker(\delta^\Sigma_n)$ and c_n for the dimension of the field of constants of $\mathcal{K}_n$ over the field of constants of k (so that c_n divides m^Σ_n). Then for each natural number n, there are natural exact commutative diagrams of R_n-modules

$$\begin{array}{ccccccccc} 0 \to & \ker(\delta^\Sigma_n) & \xrightarrow{\subseteq} & B_\Sigma(\mathcal{K}_n) & \xrightarrow{\delta^\Sigma_n} & \mathbb{Z} \to & \mathbb{Z}/(\mathbb{Z}\cdot m^\Sigma_n) & \to 0 \\ & {\scriptstyle\alpha'_n}\downarrow & & {\scriptstyle\alpha_n}\downarrow & & \Vert & {\scriptstyle\beta_n}\downarrow & \\ 0 \to & \ker(\delta^\Sigma_{n-1}) & \xrightarrow{\subseteq} & B_\Sigma(\mathcal{K}_{n-1}) & \xrightarrow{\delta^\Sigma_{n-1}} & \mathbb{Z} \to & \mathbb{Z}/(\mathbb{Z}\cdot m^\Sigma_{n-1}) & \to 0. \end{array}$$

$$\begin{array}{ccccccccccc} 0 \to & \mathcal{O}^\times_{\mathcal{K}_n,\Sigma} & \xrightarrow{\Delta^\Sigma_n} & \ker(\delta^\Sigma_n) & \to & A_\emptyset(\mathcal{K}_n) & \to & A_\Sigma(\mathcal{K}_n) & \to & (\mathbb{Z}\cdot c_n)/(\mathbb{Z}\cdot m^\Sigma_n) \to 0 \\ & {\scriptstyle\mathrm{N}_n}\downarrow & & {\scriptstyle\alpha'_n}\downarrow & & {\scriptstyle\mathrm{N}_n}\downarrow & & {\scriptstyle\mathrm{N}_n}\downarrow & & {\scriptstyle\beta'_n}\downarrow \\ 0 \to & \mathcal{O}^\times_{\mathcal{K}_{n-1},\Sigma} & \xrightarrow{\Delta^\Sigma_{n-1}} & \ker(\delta^\Sigma_{n-1}) & \to & A_\emptyset(\mathcal{K}_{n-1}) & \to & A_\Sigma(\mathcal{K}_{n-1}) & \to & (\mathbb{Z}\cdot c_{n-1})/(\mathbb{Z}\cdot m^\Sigma_{n-1}) \to 0 \end{array} \tag{19}$$

Here $G_{\mathcal{K}/k}$ acts trivially on $\mathbb{Z}$ and on each quotient $(\mathbb{Z}\cdot c_n)/(\mathbb{Z}\cdot m_n^\Sigma)$ and $\mathbb{Z}/(\mathbb{Z}\cdot m_n^\Sigma)$, the maps α_n and α'_n are induced by sending place $w \in \Sigma_{\mathcal{K}_n}$ to $f_v \cdot v$, where v is the restriction of w to $\mathcal{K}_{n-1}$ and f_v its residue degree in $\mathcal{K}_n/\mathcal{K}_{n-1}$ and the maps β_n and β'_n are induced by the inclusions $\mathbb{Z}\cdot c_n \subseteq \mathbb{Z}\cdot c_{n-1}$ and $\mathbb{Z}\cdot m_n^\Sigma \subseteq \mathbb{Z}\cdot m_{n-1}^\Sigma$. (The exact rows of the second diagram can be derived, for example, by applying the Snake lemma to the diagram of [23, (11)] and then using the isomorphism of [23, (12)], and the commutativity of both diagrams is easy to check.)

If one tensors the above diagrams with $\mathbb{Z}_p$ and then passes to the limit over n, exactness is preserved (since finitely-generated $\mathbb{Z}_p$-modules are compact) and so one obtains an inclusion $\varprojlim_{\alpha'_n}\ker(\delta_n^\Sigma) \subseteq \varprojlim_{\alpha'_n} B_\Sigma(\mathcal{K}_n)$ and an exact sequence of Λ-modules

$$\varprojlim_{\alpha'_n}(\mathbb{Z}_p\otimes_\mathbb{Z}\ker(\delta_n^\Sigma)) \to A^p_\emptyset(\mathcal{K}) \to A^p_\Sigma(\mathcal{K}) \to Q_p \tag{20}$$

where $Q_p := \varprojlim_{\beta'_n}(\mathbb{Z}_p\cdot c_n)/(\mathbb{Z}_p\cdot m_n^\Sigma)$ is a (possibly finite) quotient of $\mathbb{Z}_p$ (with trivial action of Γ). At this point we recall that, since $\mathcal{K}/K$ is ramified at only finitely many places, a result of Gold and Kisilevsky [16, Th. 2] implies $A^p_\emptyset(\mathcal{K})$ is a finitely-generated, torsion Λ-module. The last displayed seqeunce therefore implies $A^p_\Sigma(\mathcal{K})$ is a finitely-generated, torsion Λ-module and, by Proposition 4.3(ii), it then follows directly that $A_\Sigma(\mathcal{K})$ is a torsion R-module, as claimed in (i).

In addition, since $A^p_\Sigma(\mathcal{K})_{(n)}$ is the quotient of $A^p_\Sigma(\mathcal{K})$ by $\varpi_n(A^p_\Sigma(\mathcal{K}))$ (cf. [7, Lem. 3.1(i)]), a general property of finitely-generated Λ-modules implies that the natural map $A^p_\Sigma(\mathcal{K}) \to \varprojlim_n A^p_\Sigma(\mathcal{K})_{(n)}$ is bijective, thereby showing that $A^p_\Sigma(\mathcal{K})$ is pro-discrete.

For $n\in\mathbb{N}$, we write $\eta_{n,\Sigma} : A'_\Sigma(\mathcal{K}_n) \to A'_\Sigma(\mathcal{K}_{n-1})$ and $\overline{\eta}_{n,\Sigma} : R_{n-1}\otimes_{R_n} A'_\Sigma(\mathcal{K}_n) \to A'_\Sigma(\mathcal{K}_{n-1})$ for the maps induced by N_n. Then, to prove $A'_\Sigma(\mathcal{K})$ is pro-discrete, it is enough for us to show that each $\overline{\eta}_n$ is bijective. Writing ι_n for the inflation map $A'_\Sigma(\mathcal{K}_{n-1}) \to A'_\Sigma(\mathcal{K}_n)$, the composite $\eta_{n,\Sigma}\circ\iota_n$ is equal to multiplication-by-p and hence, since $|A'_\Sigma(\mathcal{K}_{n-1})|$ is prime-to-p, we deduce ι_n is injective and $\eta_{n,\Sigma}$, and hence also $\overline{\eta_{n,\Sigma}}$, is surjective. Thus, if we use ι_n to regard $A'_\Sigma(\mathcal{K}_{n-1})$ as a submodule of $A'_\Sigma(\mathcal{K}_n)$, then $\eta_{n,\Sigma}$ is induced by the action of the element $T_n := \sum_{g\in G_{\mathcal{K}_n/\mathcal{K}_{n-1}}} g$ of R_n. In particular, since $|A'_\Sigma(\mathcal{K}_n)|$ is prime-to-p (and hence cohomologically-trivial as a Γ_n-module), one has $\ker(\eta_{n,\Sigma}) = \varpi_{n-1}(A'_\Sigma(\mathcal{K}_n))$ and so $\overline{\eta}_{n,\Sigma}$ is bijective, as required. This completes the proof of (i).

Noting the above argument shows that $A^p_\Sigma(\mathcal{K})$ and $A^\ell_\Sigma(\mathcal{K})$ for $\ell\neq p$ are respectively the limits of the pro-discrete systems $(A^p_\Sigma(\mathcal{K})_{(n)})_n$ and $(A^\ell_\Sigma(\mathcal{K}_n))_n$, Proposition 2.3(i) reduces the proof of (ii) and (iii) to identifying conditions under which the respective quantities $\mu_{R_n}(A^p_\Sigma(\mathcal{K})_{(n)})$ and $\mu_{R_n}(A^\ell_\Sigma(\mathcal{K}_n))$ are finite and bounded independently of n.

To prove (ii) we note that, since $A^p_\Sigma(\mathcal{K})$ is finitely-generated over Λ, each module $A^p_\Sigma(\mathcal{K})_{(n)}$ is finitely-generated over $\Lambda_{(n)} = \mathbb{Z}_p\otimes_\mathbb{Z} R_n$. Hence, since $\mathbb{Z}_p$ is not finitely-generated over $\mathbb{Z}$, the module $A^p_\Sigma(\mathcal{K})_{(n)}$ is finitely-generated over R_n if and only if it is finite and, if this is the case, then

$$\mu_{R_n}(A^p_\Sigma(\mathcal{K})_{(n)}) = \mu_{\Lambda_{(n)}}(A^p_\Sigma(\mathcal{K})_{(n)}) \le \mu_\Lambda(A^p_\Sigma(\mathcal{K})).$$

Claim (ii) is therefore true since the structure theory of finitely-generated Λ-modules implies $A^p_\Sigma(\mathcal{K})_{(n)}$ is finite if and only if $A^p_\Sigma(\mathcal{K})^{\varpi_n=0}$ is finite.

To prove (iii), we set $N^\ell_{n,\Sigma} := \ker(\eta_{n,\Sigma})[\ell^\infty]$ for $n\in\mathbb{N}$ and note that the above argument establishes a natural isomorphism of R_n-modules

$$A^\ell_\Sigma(\mathcal{K}_n) \cong A^\ell_\Sigma(\mathcal{K}_{n-1}) \oplus N^\ell_{n,\Sigma}. \tag{21}$$

We note next that the algebra decomposition (7) induces an isomorphism of $\mathbb{Z}[1/p]$-algebras

$$R_n[1/p] \cong R_{n-1}[1/p]\times\mathcal{O}_n[1/p], \quad x\mapsto(\rho_{n,n-1}(x),\tau_{n,n}(x)).$$

In particular, since the action of R_n on $N^{\ell}_{n,\Sigma}$ factors through $\tau_{n,n}$, the decomposition (21) implies $\mu_{R_n}\big(A^{\ell}_{\Sigma}(\mathcal{K}_n)\big)$ is the maximum of $\mu_{R_{n-1}}\big(A^{\ell}_{\Sigma}(\mathcal{K}_{n-1})\big)$ and $\mu_{\mathcal{O}_n}\big(N^{\ell}_{n,\Sigma}\big)$. By using an induction on n, these facts combine to imply, firstly, that

$$\mu_{R_n}\big(A^{\ell}_{\Sigma}(\mathcal{K}_n)\big) = \max\{\mu_{\mathcal{O}_m}\big(N^{\ell}_{m,\Sigma}\big) : 0 \le m \le n\},$$

and then also that $\mu_R(A^{\ell}_{\Sigma}(\mathcal{K}))$ is finite if and only if $\mu_{\mathcal{O}_m}\big(N^{\ell}_{n,\Sigma}\big)$ is bounded independently of n. In addition, if $\mathcal{P}^{\ell}_n$ is the set of prime ideals of $\mathcal{O}_n$ lying over ℓ, then $t_n(\ell) = |\mathcal{P}^{\ell}_n|$ and the structure theorem of finitely-generated torsion $\mathcal{O}_n$-modules implies $\mu_{\mathcal{O}_n}(N^{\ell}_{n,\Sigma})$ is the maximum $d_{n,\Sigma,\ell}$ over $\mathfrak{q} \in \mathcal{P}^{\ell}_n$ of the number of invariant factors in the decomposition of $N^{\ell}_{n,\Sigma}$ whose support is $\mathfrak{q}$. It follows that $\mu_R(A^{\ell}_{\Sigma}(\mathcal{K}))$ is finite if and only if the multiplicity $d_{n,\Sigma,\ell}$ is bounded independently of n (a similar general point is observed in [3, Prop. 4.4]).

To investigate the latter condition, we note that, for each $b \in \mathbb{N}$, and each prime ideal $\mathcal{L}$ of $\mathcal{O}_n$ above ℓ, one has

$$\mathrm{r}\big(\mathcal{O}_n/\mathcal{L}^b\big) = p^{n-1}(p-1)/t_n(\ell)$$

and hence

$$d_{n,\Sigma,\ell} \cdot p^{n-1}(p-1) = t_n(\ell)d_{n,\Sigma,\ell} \cdot p^{n-1}(p-1)/t_n(\ell) \ge \mathrm{r}\big(N^{\ell}_{n,\Sigma}\big) \ge d_{n,\Sigma,\ell} \cdot p^{n-1}(p-1)/t_n(\ell). \quad (22)$$

We now write $d^*_{n,\Sigma,\ell}$ for the maximum of $d_{m,\Sigma,\ell}$ with $0 \le m \le n$, and $t(\ell)$ for the (finite) number of places of the field $\bigcup_{n\in\mathbb{N}} \mathbb{Q}(e^{2\pi i/p^n})$ that lie above ℓ. Then the above inequality implies

$$\begin{aligned} d^*_{n,\Sigma,\ell} \cdot p^n &\ge d_{0,\Sigma,\ell} + \sum\nolimits_{1\le m\le n} t_m(\ell)d_{m,\Sigma,\ell} \cdot p^{m-1}(p-1)/t_m(\ell) \\ &\ge \sum\nolimits_{0\le m\le n} \mathrm{r}\big(N^{\ell}_{m,\Sigma}\big) \\ &= \mathrm{r}(A^{\ell}_{\Sigma}(\mathcal{K}_n)) \\ &\ge d_{n,\Sigma,\ell} \cdot p^{n-1}(p-1)/t_n(\ell) \\ &\ge t(\ell)^{-1} \cdot d_{n,\Sigma,\ell} \cdot p^{n-1}(p-1), \end{aligned} \quad (23)$$

where the final inequality is true since $t_n(\ell)$ divides $t(\ell)$. These inequalities imply that $d_{n,\Sigma,\ell}$ is bounded (as n varies) if and only if $\mathrm{r}(A^{\ell}_{\Sigma}(\mathcal{K}_n)) = O(p^n)$ as $n \to \infty$, as required to prove (iii).

To prove (iv), it suffices to assume $A_{\Sigma}(\mathcal{K})$ is finitely-generated over R and deduce that it is finitely ∞-presented. After taking account of [7, Prop. 5.1(ii)], it is therefore enough for us to show $A_{\Sigma}(\mathcal{K})$ is finitely 2-presented over R.

As a first step, we note that, since $A_{\Sigma}(\mathcal{K})$ is pro-discrete, the last assertion of Proposition 2.4(ii) (with $M = A_{\Sigma}(\mathcal{K})$) implies it is finitely-presented if and only if the quantity

$$\mu_R(A_{\Sigma}(\mathcal{K})^{\varpi_n=0}) = \max\big\{\mu_{R_n}(A^{p}_{\Sigma}(\mathcal{K})^{\varpi_n=0}), \mu_{R_n}(A'_{\Sigma}(\mathcal{K})^{\varpi_n=0})\big\}$$

is finite and bounded independently of n. In this regard, we note that [7, Lem. 3.7] (with $M = A'_{\Sigma}(\mathcal{K})$ implies the R_n-module $A'_{\Sigma}(\mathcal{K})^{\varpi_n=0}$ is isomorphic to $A'_{\Sigma}(\mathcal{K})_{(n)}$, and hence that $\mu_{R_n}((A'_{\Sigma}(\mathcal{K})^{\varpi_n=0}) \le \mu_R(A'_{\Sigma}(\mathcal{K})$. To consider $\mu_{R_n}(A^{p}_{\Sigma}(\mathcal{K})^{\varpi_n=0})$, we fix (as we may, by the structure theory of Λ-modules) an exact sequence of Λ-modules

$$0 \to M_1 \to A^{p}_{\Sigma}(\mathcal{K}) \to M_2 \oplus M'_2$$

in which M_1 is finite, M_2 is a finitely-generated torsion-free $\mathbb{Z}_p$-module and M'_2 is a finite direct sum of modules of the form $\Lambda/(p^i)$. In particular, since $(M'_2)^{\varpi_n=0}$ vanishes and $A^{p}_{\Sigma}(\mathcal{K})^{\varpi_n=0}$ is finite (by (ii) and the current assumption that $A_{\Sigma}(\mathcal{K})$ is finitely-generated over R), the above sequence induces an isomorphism $M_1^{\varpi_n=0} \cong A^{p}_{\Sigma}(\mathcal{K})^{\varpi_n=0}$. Since M_1 is finite, it is therefore clear that the quantities $\mu_{R_n}(A^{p}_{\Sigma}(\mathcal{K})^{\varpi_n=0})$ are finite and bounded independently of n, as required.

We now know $A_{\Sigma}(\mathcal{K})$ is finitely-presented, and so can fix an exact sequence of R-modules

$$R^m \xrightarrow{\psi} R^n \to A_\Sigma(\mathcal{K}) \to 0.$$

We must prove that the finitely-generated R-module $\mathrm{im}(\psi)$ is finitely-presented. To do this we note that, for each n, the above sequence induces isomorphisms

$$(R^n/\,\mathrm{im}(\psi))^{\varpi_n=0}[p^\infty] \cong A_\Sigma(\mathcal{K})^{\varpi_n=0}[p^\infty] = A^p_\Sigma(\mathcal{K})^{\varpi_n=0}[p^\infty] \cong M_1^{\varpi_n=0},$$

for the module M_1 discussed above. In particular, since M_1 is finite, we can apply the criterion of Proposition 2.4(iii) to $M = \mathrm{im}(\psi)$ in order to deduce that $\mathrm{im}(\psi)$ is finitely-presented, as required to prove (iv).

Given (i), (ii), (iii) and (iv), the proof of (v) is reduced to showing that, if $t_n(\ell)\mathrm{r}(A^\ell_\Sigma(\mathcal{K}_n))/p^n$ is bounded independently of n and $\ell \neq p$, then so is the multiplicity $d_{n,\Sigma,\ell}$. But this follows directly from the fact that the penultimate inequality in (23), implies $t_n(\ell)\mathrm{r}(A^\ell_\Sigma(\mathcal{K}_n))/p^n \geq d_{n,\Sigma,\ell}\cdot(1-1/p)$.

To prove (vi) we assume $\mathcal{K} = k^{\mathrm{con}}$. In this case, every place in Σ has an open decomposition group in $G_{\mathcal{K}/k}$ and so the terms c_n and m_n^Σ occurring in (19) satisfy $c_{n+1} = pc_n$ and $m^\Sigma_{n+1} = pm^\Sigma_n$ for all sufficiently large n. This implies, in particular, that the limit module Q_p in (20) is finite. One checks similarly in this case that the limit $\varprojlim_{\alpha'_n}(\mathbb{Z}_p\otimes_\mathbb{Z} B_\Sigma(\mathcal{K}_n))$, and hence also its submodule $\varprojlim_{\alpha'_n}(\mathbb{Z}_p\otimes_\mathbb{Z}\ker(\delta^\Sigma_n))$ vanishes, and so (20) gives an exact sequence

$$0 \to A^p_\emptyset(\mathcal{K}) \to A^p_\Sigma(\mathcal{K}) \to Q_p$$

in which Q_p is finite. To show that each module $A^p_\Sigma(\mathcal{K})^{\varpi_n=0}$ is finite, we are therefore reduced to considering the special case that $\Sigma = \emptyset$.

To deal with this case, we write κ for the constant field of k. Then k is the function field of the curve C_k over κ and, if J is the Jacobian of C_k, then for each $n \in \mathbb{N}_0$ one has $A_\emptyset(\mathcal{K}_n) = J(\kappa_n)$, where κ_n is the (unique) field extension of κ of degree n. It follows that $A_\emptyset(\mathcal{K}_n) = A_\emptyset(\mathcal{K}_m)^{\varpi_n=0}$ for every $m > n$, and hence that

$$A^p_\emptyset(\mathcal{K})^{\varpi_n=0} = \big(\varprojlim_{m>n}A^p_\emptyset(\mathcal{K}_n)\big)^{\varpi_n=0} = \varprojlim_{m>n}A^p_\emptyset(\mathcal{K}_m)^{\varpi_n=0} = \varprojlim_{m>n}A^p_\emptyset(\mathcal{K}_n) = 0,$$

where the transition morphisms in the third limit are multiplication-by-p. In this case, therefore, the condition on $A^p_\emptyset(\mathcal{K})$ stated in (v) is clearly satisfied.

We now fix a prime $\ell \neq p$. Then by tensoring the upper row of (19) with $\mathbb{Z}_\ell$ one obtains an exact sequence $A^\ell_\emptyset(\mathcal{K}_n) \to A^\ell_\emptyset(\mathcal{K}_n) \to (\mathbb{Z}_\ell\cdot c_n)/(\mathbb{Z}_\ell\cdot m^\Sigma_n)$ and hence an inequality

$$\mathrm{r}\big(A^\ell_\Sigma(\mathcal{K}_n)\big) \leq \mathrm{r}\big(A^\ell_\emptyset(\mathcal{K}_n)\big) + \mathrm{r}\big((\mathbb{Z}_\ell\cdot c_n)/(\mathbb{Z}_\ell\cdot m^\Sigma_n)\big) = \mathrm{r}\big(A^\ell_\emptyset(\mathcal{K}_n)\big) + 1.$$

In addition, if $g(C_k)$ is the genus of C_k, then [24, Th. 11.12] implies that, for every n, one has

$$\mathrm{r}\big(A^\ell_\emptyset(\mathcal{K}_n)\big) = \mathrm{r}\big(J(\kappa_n)[\ell^\infty]\big) \leq 2g(C_k),$$

and hence also $\mathrm{r}\big(A^\ell_\Sigma(\mathcal{K}_n)\big) \leq 2g(C_k)+1$. (A similar observation about ℓ-ranks in this context is also made by Washington in [33, §II]).

It follows that $t_n(\ell)\mathrm{r}\big(A^\ell_\Sigma(\mathcal{K}_n)\big)/p^n \leq 2g(C_k)+1$ and hence that, in this case, the condition on $\mathrm{r}\big(A^\ell_\emptyset(\mathcal{K}_n)\big)$ stated in (v) is satisfied by the bound $2g(C_k)+1$ (which is independent of both n and ℓ). This completes the proof of Theorem 5.1. □

Remark 5.2. The condition that $A^p_\Sigma(\mathcal{K})^{\varpi_n=0}$ is finite for every n is equivalent to requiring that the characteristic polynomial of $A^p_\Sigma(\mathcal{K})$ as a (finitely-generated, torsion) R^p-module is not divisible by any cyclotomic polynomial $\Phi_i(\gamma)$ (see the proof of Proposition 3.4).

Remark 5.3. The condition on $\mathrm{r}(A^\ell_\Sigma(\mathcal{K}_n))$ in Theorem 5.1(v) implies the finite generation of $A'_\Sigma(\mathcal{K})$ over R, but may not be necessary for this. To be specific, the above argument shows that the finite generation of $A'_\Sigma(\mathcal{K})$ is equivalent to requiring the multiplicities $d_{n,\Sigma,\ell}$ are bounded independently of both n and $\ell \neq p$, and the inequalities (22) show that this condition is intrinsically linked to the ranks $\mathrm{r}\big(N^{\ell}_{n,\Sigma}\big) = \mathrm{r}\big(A^{\ell}_{\Sigma}(\mathcal{K}_n)\big) - \mathrm{r}\big(A^{\ell}_{\Sigma}(\mathcal{K}_{n-1})\big)$.

Remark 5.4. The finite generation of $A_\Sigma(\mathcal{K})$ as an R-module does not imply that the submodule $\bigoplus_{\ell\in\mathcal{P}} A^{\ell}_{\Sigma}(\mathcal{K})$ is also finitely-generated. In particular, the latter module cannot be finitely-generated if $A^{\ell}_{\Sigma}(\mathcal{K})$ is non-zero for infinitely many ℓ and such examples arise with $\mathcal{K} = k^{\mathrm{con}}$ and $\Sigma = \emptyset$ (cf. Rosen [24, Cor. to Th. 11.6]).

6. Derived Weil-étale cohomology

6.1. **Definition and basic properties.** We use the same notation as in §5. For any scheme Y of finite type over the constant field κ of k, we also write Y_{W} for the Weil-étale site on Y defined by Lichtenbaum in [21, §2]. By a slight abuse of notation, we denote by $\mathbb{G}_m$ the Weil-étale sheaf on Y that is obtained by restricting the étale sheaf $\mathbb{G}_m$ on Y to the Weil-étale site.

We assume $\Sigma \neq \emptyset$ and also fix a finite set of places Σ' of k disjoint from Σ. We write C^{Σ}_F for the affine curve $\mathrm{Spec}(\mathcal{O}_{F,\Sigma})$ and $j^{\Sigma}_F : C^{\Sigma}_F \to C_F$ for the corresponding open immersion of κ-schemes. We identify each $w \in \Sigma'_F$ with the corresponding (closed) point of C^{Σ}_F, write $\kappa(w)$ for its residue field and set $Z_w := \mathrm{Spec}(\kappa(w))$. The corresponding closed immersion $i_w : Z_w \to C^{\Sigma}_F$ induces a morphism of sheaves $\mathbb{G}_m \to i_{w,*}i^*_w\mathbb{G}_m$ on $(C^{\Sigma}_F)_{\mathrm{W}}$ and hence a morphism in $\mathsf{D}(\mathbb{Z}[G])$

$$\varrho_w : \mathsf{R}\Gamma((C^{\Sigma}_F)_{\mathrm{W}}, \mathbb{G}_m) \to \mathsf{R}\Gamma((C^{\Sigma}_F)_{\mathrm{W}}, i_{w,*}i^*_w\mathbb{G}_m) \cong \mathsf{R}\Gamma(\kappa(w)_{\mathrm{W}}, \mathbb{G}_m).$$

Following [8, §2.2], we then define the Σ'-modified Weil-étale cohomology complex $\mathsf{R}\Gamma^{\Sigma'}((C^{\Sigma}_F)_{\mathrm{W}}, \mathbb{G}_m)$ of $\mathbb{G}_m$ on $(C^{\Sigma}_F)_{\mathrm{W}}$ via the exact triangle in $D(\mathbb{Z}[G])$

$$\mathsf{R}\Gamma^{\Sigma'}((C^{\Sigma}_F)_{\mathrm{W}}, \mathbb{G}_m) \to \mathsf{R}\Gamma((C^{\Sigma}_F)_{\mathrm{W}}, \mathbb{G}_m) \xrightarrow{(\varrho_w)_w} \bigoplus_{w\in\Sigma'_F} \mathsf{R}\Gamma(\kappa(w)_{\mathrm{W}}, \mathbb{G}_m) \to \cdot \tag{24}$$

We now fix a $\mathbb{Z}_p$-extension $\mathcal{K}$ of k in which only finitely many places ramify, and impose the following conditions on F, Σ and Σ'.

Hypothesis 6.1.
 (i) *F is a finite abelian extension of k that is disjoint from $\mathcal{K}$. We set $H := G_{F/k}$.*
 (ii) *Σ is finite, non-empty and contains all places that ramify in $\mathcal{F} := F \cdot \mathcal{K}$ but no place that splits completely in $\mathcal{K}$.*
 (iii) *Σ' is finite and non-empty.*

We then regard F (and hence $\mathcal{F}$), Σ and Σ' as fixed and, for $n \in \mathbb{N}_0$, set

$$U_n := \mathcal{O}^{\times,\Sigma'}_{\mathcal{F}_n,\Sigma},\ A_n := A^{\Sigma'}_{\Sigma}(\mathcal{F}_n),\ B_n := B^0_{\Sigma}(\mathcal{F}_n)\ \text{ and }\ C^\bullet_n := \mathsf{R}\Gamma^{\Sigma'}((C^{\Sigma}_{\mathcal{F}_n})_{\mathrm{W}}, \mathbb{G}_m).$$

Here $\mathcal{O}^{\times,\Sigma'}_{\mathcal{F}_n,\Sigma}$ denotes the (finite index) submodule of $\mathcal{O}^{\times}_{\mathcal{F}_n,\Sigma}$ comprising elements that are congruent to 1 modulo all places in $\Sigma'_{\mathcal{F}_n}$ and $A^{\Sigma'}_{\Sigma}(\mathcal{F}_n)$ the ray class group of $\mathcal{O}_{\mathcal{F}_n,\Sigma}$ modulo $\Sigma'_{\mathcal{F}_n}$.

We identify $G_{\mathcal{K}/k}$ with $\Gamma = \mathbb{Z}_p$ (as in §5). Then, for each $n \in \mathbb{N}$, Hypothesis 6.1(i) identifies $G_n := G_{\mathcal{F}_n/k}$ with $H \times \Gamma_n$. In addition, for each $n \in \mathbb{N}$, Hypothesis 6.1(ii) ensures Σ contains all places that ramify in $\mathcal{F}_n/k$, and so the complex $C^\bullet_n$ belongs to $\mathsf{D}^{\mathrm{perf}}(\mathbb{Z}[G_n])$ and there exists a canonical 'projection formula' isomorphism in $\mathsf{D}^{\mathrm{perf}}(\mathbb{Z}[G_{n-1}])$

$$\theta_n : \mathbb{Z}[G_{n-1}] \otimes_{\mathbb{Z}[G_n]} \mathsf{R}\Gamma^{\Sigma'}((C^{\Sigma}_{\mathcal{F}_n})_{\mathrm{W}}, \mathbb{G}_m) \cong \mathsf{R}\Gamma^{\Sigma'}((C^{\Sigma}_{\mathcal{F}_{n-1}})_{\mathrm{W}}, \mathbb{G}_m). \tag{25}$$

(These facts are established in [6, §2.2, Lem. 1] for $\Sigma' = \emptyset$ and in [8, §2.2] for $\Sigma' \neq \emptyset$.) The associated family $(C^\bullet_n, \theta_n)_n$ satisfies Hypothesis 4.9 and so, recalling that $\mathbb{A}$ denotes $R[H] = \varprojlim_n \mathbb{Z}[G_n]$, we obtain a well-defined object of $\mathsf{D}(\mathbb{A})$ by setting

$$\mathsf{R}\Gamma^{\Sigma'}((C^{\Sigma}_{\mathcal{F}})_{\mathrm{W}}, \mathbb{G}_m) := \mathrm{R}\varprojlim_{\theta_n} C^\bullet_n.$$

Definition 6.2. In each degree i, the 'Iwasawa Weil-étale cohomology' and 'derived Iwasawa Weil-étale cohomology' of $\mathbb{G}_m$ relative to $(\mathcal{F},\Sigma,\Sigma')$ are the respective $\mathbb{A}$-modules

$$H^{i,\Sigma'}_{\mathrm{Iw}}((C^{\Sigma}_{\mathcal{F}})_{\mathrm{W}},\mathbb{G}_m) := \varprojlim_n H^{i,\Sigma'}((C^{\Sigma}_{\mathcal{F}_n})_{\mathrm{W}},\mathbb{G}_m) \;\text{ and }\; H^{i,\Sigma'}_{\mathrm{dIw}}((C^{\Sigma}_{\mathcal{F}})_{\mathrm{W}},\mathbb{G}_m) := H^i(\mathsf{R}\Gamma^{\Sigma'}((C^{\Sigma}_{\mathcal{F}})_{\mathrm{W}},\mathbb{G}_m)),$$

where the limit are taken with respect to the maps $H^i(\theta_n)$.

To compute these modules, we use the inverse limits

$$A(\mathcal{F}) := A^{\Sigma'}_{\Sigma}(\mathcal{F}) := \varprojlim_{\mathrm{N}_n} A_n,\;\; U(\mathcal{F}) = U^{\Sigma'}_{\Sigma}(\mathcal{F}) = \varprojlim_{\mathrm{N}_n} U_n \;\;\text{ and }\;\; B^0(\mathcal{F}) = B^0_{\Sigma}(\mathcal{F}) := \varprojlim_{\mu_n} B^0_n.$$

Here the transition maps $\mu_n : B^0_n \to B^0_{n-1}$ are induced by the maps $\Sigma_{\mathcal{F}_n} \to \Sigma_{\mathcal{F}_{n-1}}$ obtained by restricting places of $\mathcal{F}_n$ to $\mathcal{F}_{n-1}$.

Lemma 6.3. *For each index $* \in \{\mathrm{Iw}, \mathrm{dIw}\}$, the group $H^{i,\Sigma'}_*((C^{\Sigma}_{\mathcal{F}})_{\mathrm{W}},\mathbb{G}_m)$ vanishes for $i \notin \{0,1\}$ and $H^{0,\Sigma'}_*((C^{\Sigma}_{\mathcal{F}})_{\mathrm{W}},\mathbb{G}_m) = U(\mathcal{F})$. Further, there exist canonical short exact sequences of R-modules*

$$0 \to \varprojlim^1_{\mathrm{N}_n} U_n \to H^{1,\Sigma'}_{\mathrm{dIw}}((C^{\Sigma}_{\mathcal{F}})_{\mathrm{W}},\mathbb{G}_m) \to H^{1,\Sigma'}_{\mathrm{Iw}}((C^{\Sigma}_{\mathcal{F}})_{\mathrm{W}},\mathbb{G}_m) \to 0,$$
$$0 \to A(\mathcal{F}) \to H^{1,\Sigma'}_{\mathrm{Iw}}((C^{\Sigma}_{\mathcal{F}})_{\mathrm{W}},\mathbb{G}_m) \to B^0(\mathcal{F}) \to 0.$$

Proof. For each n, the computations of Lichtenbaum [21, Th. 6.1] (or [6, §2.2, Lem. 1]) in the case $\Sigma' = \emptyset$ and of [8, Rems. 2.5(iii) and 2.7] for $\Sigma' \neq \emptyset$, show $H^{i,\Sigma'}((C^{\Sigma}_{\mathcal{F}_n})_{\mathrm{W}},\mathbb{G}_m) = (0)$ for $i \notin \{0,1\}$ and that $H^{0,\Sigma'}((C^{\Sigma}_{\mathcal{F}_n})_{\mathrm{W}},\mathbb{G}_m) = U_n$, $H^{1,\Sigma'}((C^{\Sigma}_{\mathcal{F}_n})_{\mathrm{W}},\mathbb{G}_m)_{\mathrm{tor}} = A_n$ and $H^{1,\Sigma'}((C^{\Sigma}_{\mathcal{F}_n})_{\mathrm{W}},\mathbb{G}_m)_{\mathrm{tf}} = B^0_n$. Further, with respect to these identifications, the maps induced on cohomology by the isomorphism θ_n in (25) agree in the first two cases with those induced by the norm N_n and in the third case is the restriction map $B^0_n \to B^0_{n-1}$.

Given these descriptions, all assertions except for the second exact sequence now follow directly from the general result of Proposition 4.11(i). For the second exact sequence one only needs to note that, since each A_n is finite, the derived limit $\varprojlim^1_n A_n$ vanishes. □

We now assume to be given a pro-discrete system $(X_n)_n$, where each X_n is an $\mathbb{A}_n$-submodule of $A_n \otimes_{\mathbb{Z}} \mathbb{Z}[1/|G_n|]$, and write X for the pro-discrete $\mathbb{A}$-submodule $\varprojlim_n X_n$ of $A(\mathcal{F})$. Then, since $|X_n|$ and $|G_n|$ are coprime, the convergent spectral sequence from [32, III, 4.6.10]

$$\prod_{i\in\mathbb{Z}} \mathsf{Ext}^a_{\mathbb{A}_n}(H^i(X_n[-1]), H^{b+i}(C^\bullet_n)) \implies H^{a+b}(\mathsf{R}\mathrm{Hom}_{\mathbb{A}_n}(X_n[-1], C^\bullet_n)) \tag{26}$$

collapses to imply bijectivity of the map $\mathsf{R}\mathrm{Hom}_{\mathsf{D}(\mathbb{A}_n)}(X_n[-1], C^\bullet_n) \to \mathrm{Hom}_{\mathbb{A}_n}(X_n, \mathcal{X}^1_n)$ that sends ϕ to $H^1(\phi)$. We write ι_{X_n} for the unique morphism $X_n[-1] \to C^\bullet_n$ in $\mathsf{D}(\mathbb{A}_n)$ for which $H^1(\iota_{X_n})$ is the inclusion $X_n \subseteq A_n \subseteq \mathcal{X}^1_n$, and $C^\bullet_{X_n}$ for the mapping cone of ι_{X_n}. We then obtain a commutative diagram in $\mathsf{D}(\mathbb{A}_{n-1})$

$$\begin{tikzcd}
\mathbb{A}_{n-1} \otimes^{\mathsf{L}}_{\mathbb{A}_n} X_n[-1] \arrow[r, "1\otimes_{\mathbb{A}_n}\iota_{X_n}"] \arrow[d, "\theta_{X,n}"'] & \mathbb{A}_{n-1} \otimes^{\mathsf{L}}_{\mathbb{A}} C^\bullet_n \arrow[r] \arrow[d, "\theta_n"'] & \mathbb{A}_{n-1} \otimes^{\mathsf{L}}_{\mathbb{A}_n} C^\bullet_{X_n} \arrow[r] \arrow[d, "\theta'_{X,n}"'] & \cdot \\
X_{n-1}[-1] \arrow[r, "\iota_{X_{n-1}}"] & C^\bullet_{n-1} \arrow[r] & C^\bullet_{X_{n-1}} \arrow[r] & \cdot
\end{tikzcd} \tag{27}$$

Here the upper row is the exact triangle obtained by applying the exact functor $\mathbb{A}_{n-1} \otimes^{\mathsf{L}}_{\mathbb{A}_n} -$ to the tautological exact triangle arising from the definition of mapping cone and the lower row is the corresponding exact triangle with n replaced by $n-1$. In addition, the morphism $\theta_{X,n}$ is induced by the given isomorphism $(X_n)_{(n-1)} \cong X_{n-1}$ and is an isomorphism in $\mathsf{D}(\mathbb{A}_{n-1})$. In particular, since θ_n is also an isomorphism in $\mathsf{D}(\mathbb{A}_{n-1})$ and the first square commutes, the Octahedral axiom implies

$\theta'_{X,n}$ is an isomorphism in $\mathsf{D}(\mathbb{A}_{n-1})$. It follows that the family $(C^\bullet_{X_n}, \theta'_{X,n})_n$ satisfies Hypothesis 4.9 and so we obtain a well-defined object of $\mathsf{D}(\mathbb{A})$ by setting

$$C^\bullet_X := \mathrm{R}\varprojlim_{\theta'_{X,n}} C^\bullet_{X_n}.$$

Lemma 6.4. *If $i \notin \{0,1\}$, then $H^i(C^\bullet_X)$ vanishes. In addition, $H^0(C^\bullet_X) = U(\mathcal{F})$ and there exist canonical short exact sequences of $\mathbb{A}$-modules*

$$0 \to X \to H^{1,\Sigma'}_{\mathrm{dIw}}((C^\Sigma_{\mathcal{F}})_{\mathrm{W}}, \mathbb{G}_m) \to H^1(C^\bullet_X) \to 0$$
$$0 \to A(\mathcal{F})/X \xrightarrow{\subseteq} \varprojlim_n H^1(C^\bullet_{X_n}) \to B^0(\mathcal{F}) \to 0$$
$$0 \to \varprojlim{}^1_{\mathrm{N}_n} U_n \to H^1(C^\bullet_X) \to \varprojlim_n H^1(C^\bullet_{X_n}) \to 0.$$

Proof. With $\theta_{X,n}$ the isomorphism in $\mathsf{D}(\mathbb{A}_{n-1})$ that occurs in (27), the explicit definition (16) of the derived limit $\mathrm{R}\varprojlim_{\theta_{X,n}} X_n[-1]$ implies that it identifies with the complex $X[-1]$. Hence, upon combining the morphisms of exact triangles that correspond to (27) with the general result of Proposition 4.11(ii), one obtains an exact triangle in $\mathsf{D}(\mathbb{A})$

$$X[-1] \xrightarrow{\iota_X} \mathsf{R}\Gamma^{\Sigma'}((C^\Sigma_{\mathcal{F}})_{\mathrm{W}}, \mathbb{G}_m) \to C^\bullet_X \to \cdot$$

in which $H^1(\iota_X)$ is the inclusion map $X \subseteq \mathcal{X}^1$. All assertions except for the second and third claimed exact sequences now follow by combining the long exact cohomology sequence of this triangle together with the descriptions of the cohomology of $\mathsf{R}\Gamma^{\Sigma'}((C^\Sigma_{\mathcal{F}})_{\mathrm{W}}, \mathbb{G}_m)$ given in Lemma 6.3.

Next we note that the long exact cohomology sequence of the lower row of (27) (with $n-1$ replaced by n) implies $C^\bullet_{X_n}$ is acyclic outside degrees 0 and 1, that $H^0(C^\bullet_{X_n}) = H^0(C^\bullet_n) = U_n$ and, after taking account of the description of $H^1(C^\bullet_n)$ given in the proof of Lemma 6.3, that there exists a canonical short exact sequence $0 \to A_n/X_n \to H^1(C^\bullet_{X_n}) \to B^0_n \to 0$.

The second claimed exact sequence is obtained by passing to the inverse limit over n of the latter sequences and noting that, since each module A_n and X_n is finite, exactness is preserved and also $\varprojlim_n (A_n/X_n) = A(\mathcal{F})/X$. Finally, the third claimed exact sequence follows from the general result of Proposition 4.11(i) and the fact that $H^0(C^\bullet_n) = U_n$ for each n. □

6.2. **Iwasawa theory over $\mathbb{Z}[[\mathbb{Z}_p]]$.** For each pro-discrete system $X = (X_n)_n$ as above, we now investigate the $\mathbb{A}$-module $H^1(C^\bullet_X) = H^{1,\Sigma'}_{\mathrm{dIw}}((C^\Sigma_{\mathcal{F}})_{\mathrm{W}}, \mathbb{G}_m)/X$.

To help the reader, we recall that an ideal I of $\mathbb{A}_n = \mathbb{Z}[G_n]$ is said to be 'invertible' if it is locally-free of rank 1 as a $\mathbb{A}_n$-module and that, for any such ideal, the set

$$I^{-1} := \{x \in \mathbb{Q}[G_n] : x \cdot I \subseteq \mathbb{A}_n\}$$

is a locally-free $\mathbb{A}_n$-module of rank 1 such that $I \cdot I^{-1} = \mathbb{A}_n$ (cf. [13, §35]). By a slight abuse of notation, in the sequel we also write $\varrho_{n,n-1}$ for the $\mathbb{Q}$-linear extension $\mathbb{Q}[G_n] \to \mathbb{Q}[G_{n-1}]$ of $\varrho_{n,n-1}$.

6.2.1. *Statement of the main results.* For $n \in \mathbb{N}_0$, we write $\theta_n = \theta^{\Sigma'}_{\mathcal{F}_n,\Sigma}$ for the (Σ, Σ')-relative Stickelberger element of $\mathcal{F}_n/k$. We recall θ_n is an element of $\mathbb{A}_n$ that is explicitly defined, for example, in [8, §5.1] and interpolates the values at zero of the Σ'-modified Σ-truncated Dirichlet L-series attached to complex characters of $G_{\mathcal{F}_n/k}$. In particular, the inflation invariance of L-series implies that, for each $n \in \mathbb{N}$, the natural map $\varrho_{n,n-1} : \mathbb{A}_n \to \mathbb{A}_{n-1}$ sends θ_n to θ_{n-1} and so one obtains a well-defined Iwasawa-theoretic Stickelberger element by setting

$$\theta_{\mathcal{F}} = \theta^{\Sigma'}_{\mathcal{F},\Sigma} := (\theta_n)_n \in \varprojlim_{\varrho_{n,n-1}} \mathbb{A}_n = \mathbb{A}.$$

Our aim is to explicitly relate these elements to the structure of $\mathbb{A}$-modules arising from the derived Weil-étale cohomology of $\mathbb{G}_m$.

In order to state our main result in this regard, we note that, since X_n is finite and of order prime to $|G_n|$, the $\mathbb{A}_n$-ideal $\mathrm{Fitt}^0_{\mathbb{A}_n}(X_n)$ is invertible. We further recall that, since $(X_n, \mathrm{N}_n)_n$ is a pro-discrete system, the argument of Proposition 4.3(ii) implies, for $n \in \mathbb{N}$, that $\varrho_{n,n-1}(\mathrm{Fitt}^0_{\mathbb{A}_n}(X_n)) = \mathrm{Fitt}^0_{\mathbb{A}_{n-1}}(X_{n-1})$. In particular, since $\varrho_{n,n-1}(\theta_n) = \theta_{n-1}$ for $n \in \mathbb{N}$, it follows that each of the families

$$\big(\mathrm{Fitt}^0_{\mathbb{A}_n}(X_n)^{-1}, \varrho_{n,n-1}\big)_n,\quad \big(\theta_n \cdot \mathrm{Fitt}^0_{\mathbb{A}_n}(X_n)^{-1}, \varrho_{n,n-1}\big)_n \quad \text{and} \quad \big(\mathrm{Ann}_{\mathbb{A}_n}(\theta_n), \varrho_{n,n-1}\big)_n$$

is an inverse system.

Theorem 6.5. *Assume Hypothesis 6.1 and also that $H^{1,\Sigma'}_{\mathrm{Iw}}((C^{\Sigma}_{\mathcal{F}})_{\mathrm{W}}, \mathbb{G}_m)/X$ is finitely-generated over $\mathbb{Z}[\![\mathbb{Z}_p]\!]$. Then the $\mathbb{A}$-module $H^{1,\Sigma'}_{\mathrm{dIw}}((C^{\Sigma}_{\mathcal{F}})_{\mathrm{W}}, \mathbb{G}_m)/X$ is finitely-presented and torsion and there exist canonical short exact sequences of $\mathbb{A}$-modules*

$$0 \to \mathrm{Fit}^0_{\mathbb{A}}(H^{1,\Sigma'}_{\mathrm{dIw}}((C^{\Sigma}_{\mathcal{F}})_{\mathrm{W}}, \mathbb{G}_m)/X) \xrightarrow{\subseteq} \varprojlim_n(\theta_n \cdot \mathrm{Fit}^0_{\mathbb{A}_n}(X_n)^{-1}) \to \varprojlim{}^1_n \mathrm{Ann}_{\mathbb{A}_n}(\theta_n) \to 0$$

and

$$0 \to \theta_{\mathcal{F}} \cdot \varprojlim_n \mathrm{Fit}^0_{\mathbb{A}_n}(X_n)^{-1} \to \varprojlim_n(\theta_n \cdot \mathrm{Fit}^0_{\mathbb{A}_n}(X_n)^{-1}) \to \varprojlim{}^1_n \mathrm{Ann}_{\mathbb{A}_n}(\theta_n) \to 0$$

in which all limits are defined with respect to the projection maps $\varrho_{n,n-1}$. In particular, one has

$$(\theta_{\mathcal{F}})^2 \in \mathrm{Fit}^0_{\mathbb{A}}(H^{1,\Sigma'}_{\mathrm{dIw}}((C^{\Sigma}_{\mathcal{F}})_{\mathrm{W}}, \mathbb{G}_m)/X). \tag{28}$$

Further, the derived limit $\varprojlim^1_n \mathrm{Ann}_{\mathbb{A}_n}(\theta_n)$ is divisible and p-torsion-free and vanishes only if $|\Sigma| = 1$ and the unique place in Σ has full decomposition group in $G_{\mathcal{F}/k}$.

This result will be proved in §6.2.2 and the finite generation of $H^{1,\Sigma'}_{\mathrm{Iw}}((C^{\Sigma}_{\mathcal{F}})_{\mathrm{W}}, \mathbb{G}_m)/X$ will be discussed in §6.2.4.

The following result records a consequence of the proof of Theorem 6.5 concerning the finite-presentability of $H^{1,\Sigma'}_{\mathrm{Iw}}((C^{\Sigma}_{\mathcal{F}})_{\mathrm{W}}, \mathbb{G}_m)/X$. In particular, it shows that, even if the 'obvious' Iwasawa-theoretic Weil-étale cohomology group $H^{1,\Sigma'}_{\mathrm{Iw}}((C^{\Sigma}_{\mathcal{F}})_{\mathrm{W}}, \mathbb{G}_m)$ is finitely-generated over R, it is very unlikely to be finitely-presented and so cannot be used in any general main-conjecture-type formalism relative to $\mathbb{A}$. (This observation motivates our general use of derived Weil-étale cohomology.)

Corollary 6.6. *Assume Hypothesis 6.1 and also that $H^{1,\Sigma'}_{\mathrm{Iw}}((C^{\Sigma}_{\mathcal{F}})_{\mathrm{W}}, \mathbb{G}_m)/X$ is finitely-generated as an R-module. Then the following conditions are equivalent.*

(i) *$H^{1,\Sigma'}_{\mathrm{Iw}}((C^{\Sigma}_{\mathcal{F}})_{\mathrm{W}}, \mathbb{G}_m)/X$ is finitely-presented.*
(ii) *The natural map $H^{1,\Sigma'}_{\mathrm{dIw}}((C^{\Sigma}_{\mathcal{F}})_{\mathrm{W}}, \mathbb{G}_m) \to H^{1,\Sigma'}_{\mathrm{Iw}}((C^{\Sigma}_{\mathcal{F}})_{\mathrm{W}}, \mathbb{G}_m)$ is an isomorphism.*
(iii) *The inverse system $(U_n, \mathrm{N}_n)_n$ is Mittag-Leffler.*
(iv) *$|\Sigma| = 1$ (so that $\mathcal{F}/k$ is ramified at at most one place) and the unique place in Σ has full decomposition subgroup in $G_{\mathcal{F}/k}$.*

This result will be proved in §6.2.3

6.2.2. *The proof of Theorem 6.5.* As a first step, we show that the family $(C^{\bullet}_{X_n})_n$ satisfies the conditions of Proposition 4.11(iv) with $a = 0$.

The validity of condition (a) in this case follows by combining the exact triangle given by the lower row of (27) together with the fact that $C^{\bullet}_{n-1}$ belongs to $\mathsf{D}^{\mathrm{perf}}(\mathbb{A}_{n-1})$ and that the G_{n-1}-module X_{n-1} is finite and cohomologically-trivial (since its order is prime to $|G_n|$). To check the remaining conditions, we use the explicit descriptions of the groups $H^i(C^{\bullet}_{X_n})$ given in the proof of Lemma 6.4.

In particular, to verify (b), we need to show U_n is torsion-free. However, if $u \in U_n \setminus \{1\}$ satisfies $u^m = 1$ for some $m \in \mathbb{N}$, then, modulo any place in Σ', there is a congruence

$$m \equiv \sum\nolimits_{a=0}^{a=m-1} u^a = (u^m - 1)/(u - 1) = 0$$

which contradicts the fact that m is prime to the characteristic p of $\mathcal{F}_n$. Similarly, (c) is valid since the Riemann-Roch Theorem implies that the map $U_n \to B_n^0$ sending u to $(\mathrm{ord}_w(u)\cdot \deg_\kappa(w))_{w\in\Sigma_{\mathcal{F}_n}}$ induces a composite isomorphism of $\mathbb{Q}[G_n]$-modules

$$\mathbb{Q}\otimes_{\mathbb{Z}} H^0(C^\bullet_{X_n}) = \mathbb{Q}\otimes_{\mathbb{Z}} U_n \cong \mathbb{Q}\otimes_{\mathbb{Z}} B_n^0 \cong \mathbb{Q}\otimes_{\mathbb{Z}} H^1(C^\bullet_{X_n}) \tag{29}$$

(in which the second isomorphism is induced by the fact that A_n/X_n is finite).

Finally, we note that $\varprojlim_n H^1(C^\bullet_{X_n})$ is equal to $H^{1,\Sigma'}_{\mathrm{Iw}}((C^\Sigma_{\mathcal{F}})_{\mathrm{W}},\mathbb{G}_m)/X$ and so is, by explicit assumption, finitely-generated over R, as required to verify (d).

Having verified all hypotheses of Proposition 4.11(iv), we can therefore now deduce from its conclusion that $H^{1,\Sigma'}_{\mathrm{dIw}}((C^\Sigma_{\mathcal{F}})_{\mathrm{W}},\mathbb{G}_m)/X$ is a finitely-presented $\mathbb{A}$-module that verifies the hypotheses of Proposition 4.5 and, in addition, that the natural map

$$\big(H^{1,\Sigma'}_{\mathrm{dIw}}((C^\Sigma_{\mathcal{F}})_{\mathrm{W}},\mathbb{G}_m)/X\big)_{(n)} \to H^1(C^\bullet_{X_n}) = H^{1,\Sigma'}((C^\Sigma_{\mathcal{F}_n})_{\mathrm{W}},\mathbb{G}_m)/X_n$$

is bijective. Then, from Proposition 4.5, we can deduce the existence of a canonical short exact sequence

$$0\to \mathrm{Fit}^0_{\mathbb{A}}\big(H^{1,\Sigma'}_{\mathrm{dIw}}((C^\Sigma_{\mathcal{F}})_{\mathrm{W}},\mathbb{G}_m)/X\big) \xrightarrow{\subseteq} \varprojlim_n \mathrm{Fit}^0_{\mathbb{A}_n}\big(C^\bullet_{X_n}\big) \to \varprojlim{}^1_n \mathrm{Ann}_{\mathbb{A}_n}\big(\mathrm{Fit}^0_{\mathbb{A}_n}\big(C^\bullet_{X_n}\big)\big)\to 0.$$

The first displayed exact sequence in Theorem 6.5 now follows directly from the next result.

Proposition 6.7. *For each n the following claims are valid.*

(i) $\mathrm{Fit}^0_{\mathbb{A}_n}\big(C^\bullet_{X_n}\big) = \theta_n\cdot \mathrm{Fitt}^0_{\mathbb{A}_n}(X_n)^{-1}$.

(ii) $\mathrm{Ann}_{\mathbb{A}_n}\big(\theta_n\cdot \mathrm{Fitt}^0_{\mathbb{A}_n}(X_n)^{-1}\big) = \mathrm{Ann}_{\mathbb{A}_n}(\theta_n)$.

Proof. Since X_n is finite and of order prime to $|G_n|$, the natural short exact sequence of $\mathbb{A}_n$-modules

$$0\to X_n \to H^1(C^\bullet_n)\to H^1(C^\bullet_{X_n})\to 0$$

is split, and so $\mathrm{Fit}^0_{\mathbb{A}_n}\big(C^\bullet_n\big) = \mathrm{Fit}^0_{\mathbb{A}_n}\big(C^\bullet_{X_n}\big)\cdot \mathrm{Fit}^0_{\mathbb{A}_n}\big(X_n\big)$. To prove (i), we are therefore reduced to recalling the key equality

$$\mathrm{Fit}^0_{\mathbb{A}_n}\big(C^\bullet_n\big) = \theta_n\cdot \mathbb{A}_n. \tag{30}$$

Indeed, this equality follows directly by combining the results of Kurihara, Sano and the first author in [8, Th. 1.5(i) and Prop. 3.4] with the data $(K/k,S,T,V,r)$ taken to be $(\mathcal{F}_n/k,\Sigma,\Sigma',\emptyset,0)$ together with the observations in [8, Rem. 3.3(ii) and Rem. 5.3(i)].

Next we observe that to prove (ii) it suffices to verify the stated equality after tensoring with $\mathbb{Z}_\ell$ for each prime ℓ. Then we note that each $\mathbb{Z}_\ell[G_n]$-module $\mathbb{Z}_\ell\otimes_{\mathbb{Z}}\mathrm{Fitt}^0_{\mathbb{A}_n}(X_n)^{-1}$ is free of rank 1, and hence that

$$\mathbb{Z}_\ell\otimes_{\mathbb{Z}}\mathrm{Ann}_{\mathbb{A}_n}\big(\theta_n\cdot \mathrm{Fitt}^0_{\mathbb{A}_n}(X_n)^{-1}\big) = \mathrm{Ann}_{\mathbb{Z}_\ell[G_n]}(\theta_n\cdot\mathbb{Z}_\ell[G_n]) = \mathrm{Ann}_{\mathbb{Z}_\ell[G_n]}(\theta_n) = \mathbb{Z}_\ell\otimes_{\mathbb{Z}}\mathrm{Ann}_{\mathbb{A}_n}(\theta_n).$$

This proves (iii). □

To obtain the remainder of Theorem 6.5 we combine the following two results.

Lemma 6.8. *Set $I_n := \mathrm{Fit}^0_{\mathbb{A}_n}(X_n)$. Then the family $\big(I_n^{-1},\varrho_{n,n-1}\big)$ is pro-discrete, and the associated pro-discrete $\mathbb{A}$-module $J := \varprojlim_{\varrho_{n,n-1}} I_n^{-1}$ is finitely-generated.*

Proof. Since each $\mathbb{A}_n$-module I_n^{-1} is locally-free of rank 1, the Forster-Swan Theorem implies, via [7, Rem. 3.14], that $\mu_{\mathbb{A}_n}\big(I_n^{-1}\big)\le 2$. At the same time, the equality $\varrho_{n,n-1}(I_n) = I_{n-1}$ noted just before the statement of Theorem 6.5 combines with $I_n\cdot I_n^{-1} = \mathbb{A}_n$ to imply $\varrho_{n,n-1}(I_n^{-1}) = I_{n-1}^{-1}$ and hence that there exists an induced surjective map $(I_n^{-1})_{(n-1)}\to I_{n-1}^{-1}$. In particular, since I_n^{-1} is a locally-free $\mathbb{A}_n$-module, the latter map must be bijective and so J is a pro-discrete $\mathbb{A}$-module. Then, since $\mu_{\mathbb{A}_n}\big(I_n^{-1}\big)\le 2$ for every n, Proposition 2.3(i) implies that J is finitely-generated. □

Lemma 6.9. *The following claims are valid.*

(i) $\theta_{\mathcal{F}}$ *is a non-zero divisor of* $\mathbb{A}$.

(ii) *The derived limit* $\varprojlim^1_n \mathrm{Ann}_{\mathbb{A}_n}(\theta_n)$ *is divisible and* p*-torsion-free. Further, this limit vanishes if and only if* $|\Sigma| = 1$ *and the unique place in* Σ *has full decomposition subgroup in* $G_{\mathcal{F}/k}$.

Proof. For each n we set $Y_n := \mathrm{Ann}_{\mathbb{A}_n}(\theta_n)$. Then to prove both (i) and the first sentence in (ii), we are reduced, by Lemma 4.7, to proving that $\varprojlim_n \big(\mathbb{Z}_p \otimes_{\mathbb{Z}} Y_n\big)$ vanishes. To show this, we note that a comparison of the order of vanishing formula for Artin L-series proved by Tate in [29, Chap. I, Prop. 3.4] with the definition of Stickelberger elements implies that, for each $n \in \mathbb{N}$ and ψ in $\mathrm{Hom}(G_n, \mathbb{C}^\times)$, one has

$$\dim_{\mathbb{C}}\big(e_\psi(\mathbb{C} \otimes_{\mathbb{Z}} Y_n)\big) > 0 \iff e_\psi(\theta_n) = 0 \iff \dim_{\mathbb{C}}\big(e_\psi(\mathbb{C} \otimes_{\mathbb{Z}} B^0_n))\big) > 0. \tag{31}$$

It follows that Y_n is only supported on characters of G_n that are trivial on the decomposition subgroup of some place in Σ. Hence, since (the final assertion of) Hypothesis 6.1(ii) implies each such decomposition subgroup is open in $G_{\mathcal{F}/k}$, there exists $n_0 \in \mathbb{N}$ such that, for all $m > n \geq n_0$, $\varrho_{m,n}$ acts as multiplication by p^{m-n} on Y_m. It follows that $\varrho_{m,n}(Y_m) \subseteq p^{m-n} \cdot \mathbb{A}_n$ and so $\varprojlim_n \big(\mathbb{Z}_p \otimes_{\mathbb{Z}} Y_n\big)$ vanishes, as required.

To verify the second assertion in (ii) we note that, since $\mathbb{Z}_p/\mathbb{Z}$ is uniquely p-divisible, the above argument implies that $\varprojlim^1_n Y_n$ identifies with $(\mathbb{Z}_p/\mathbb{Z}) \otimes_{\mathbb{Z}} Y_m$ for any $m \geq n_0$. It follows that $\varprojlim^1_n Y_n$ vanishes if and only if Y_m vanishes for any $m \geq m_0$. In addition, by (31), one knows that Y_m vanishes if and only if B^0_m vanishes and it is easily checked this is true if and only if both $|\Sigma| = 1$ and the unique place in Σ has full decomposition subgroup in G_m. This completes the proof. □

We now set $N' := \theta_{\mathcal{F}} \cdot J$, with J as in Lemma 6.8. Then, since $N' \subseteq \varprojlim_{\varrho'_n}(\theta_n \cdot I_n^{-1})$, Proposition 6.7(ii) implies N' is an ideal of $\mathbb{A}$. In addition, Lemma 6.9(i) implies N' is isomorphic to J as a $\mathbb{A}$-module and hence, by Lemma 6.8, is a finitely-generated pro-discrete $\mathbb{A}$-module. For the same reason, each $\mathbb{A}_{(n)}$-module $N'_{(n)}$ is isomorphic to $J_{(n)}$ and hence, via the final assertion of Proposition 2.4(i), to I_n^{-1} and so is locally-free of rank 1. In addition, for each n, one has $\varrho_n(N') = \varrho_n(\theta_{\mathcal{F}}) \cdot I_n^{-1} = \theta_n \cdot I_n^{-1}$. Given these facts, the second exact sequence in Theorem 6.5 is obtained by applying Remark 4.6 with the ideal N taken to be N' and then taking account of Proposition 6.7(iii).

Next we note that

$$(\theta_{\mathcal{F}})^2 \in \theta_{\mathcal{F}} \cdot (\theta_{\mathcal{F}} \cdot J) \subseteq \theta_{\mathcal{F}} \cdot \varprojlim_n(\theta_n \cdot I_n^{-1}) \subseteq \mathrm{Fit}^0_{\mathbb{A}}\big(H^{1,\Sigma'}_{\mathrm{dIw}}((C^\Sigma_{\mathcal{F}})_{\mathrm{W}}, \mathbb{G}_m)/X\big).$$

Here the containment is true since $\mathbb{A} \subseteq J$, the first inclusion follows directly from the second exact sequence in Theorem 6.5 and the second inclusion is a consequence of the first exact sequence in Theorem 6.5 and the fact that the $\mathbb{A}$-module $\varprojlim^1_n \mathrm{Ann}_{\mathbb{A}_n}(\theta_n)$ is annihilated by $\theta_{\mathcal{F}}$ (since $\varrho_n(\theta_{\mathcal{F}}) = \theta_n$ for all n). In particular, since $\theta_{\mathcal{F}}$ is a non-zero divisor of $\mathbb{A}$ (by Lemma 6.9(i)), this displayed containment implies that $H^{1,\Sigma'}_{\mathrm{dIw}}((C^\Sigma_{\mathcal{F}})_{\mathrm{W}}, \mathbb{G}_m)/X$ is a torsion $\mathbb{A}$-module.

At this stage, to complete the proof of Theorem 6.5, it only remains for us to note that its final assertion coincides with the second sentence of Lemma 6.9(ii).

6.2.3. *The proof of Corollary 6.6.* From Lemma 6.4, there exists an exact sequence of R-modules

$$0 \to \varprojlim^1_{\mathrm{N}_n} U_n \to H^{1,\Sigma'}_{\mathrm{dIw}}((C^\Sigma_{\mathcal{F}})_{\mathrm{W}}, \mathbb{G}_m)/X \to H^{1,\Sigma'}_{\mathrm{Iw}}((C^\Sigma_{\mathcal{F}})_{\mathrm{W}}, \mathbb{G}_m)/X \to 0.$$

In particular, since Theorem 6.5 implies (under the stated hypotheses) that the central module in this sequence is finitely-presented, the general results of [15, Th. 2.1.2, (2) and (3)] imply that the R-module $H^{1,\Sigma'}_{\mathrm{Iw}}((C^\Sigma_{\mathcal{F}})_{\mathrm{W}}, \mathbb{G}_m)/X$ is finitely-presented if and only if the R-module $\varprojlim^1_{\mathrm{N}_n} U_n$ is finitely-generated.

Next we note that, since each of the (finitely many) places in Σ has an open decomposition group in $G_{\mathcal{F}/k}$, there exists a natural number m such that, for all $n \in \mathbb{N}$, the action of R on B_n factors through the projection $R \to R_m$. Since each group U_n is torsion-free, the central isomorphism in (29) then implies that the same is true of the action of R on U_n. This in turn implies that the action of R on $\varprojlim^1_{\mathrm{N}_n} U_n$ factors through R_m and so, since R_m is a finitely-generated abelian group and $\varprojlim^1_{\mathrm{N}_n} U_n$ is divisible (by Lemma 4.7), the derived limit $\varprojlim^1_{\mathrm{N}_n} U_n$ is finitely-generated over R if and only if it vanishes.

Now, since each group U_n is countable, $\varprojlim^1_{\mathrm{N}_n} U_n$ vanishes if and only if the inverse system $(U_n, \mathrm{N}_n)_n$ is Mittag-Leffler (cf. [17, p. 242, Prop.]).

In addition, the first exact sequence in Lemma 6.3 implies that $\varprojlim^1_{\mathrm{N}_n} U_n$ vanishes if and only if the natural map $H^{1,\Sigma'}_{\mathrm{dIw}}((C^{\Sigma}_{\mathcal{F}})_{\mathrm{W}}, \mathbb{G}_m) \to H^{1,\Sigma'}_{\mathrm{Iw}}((C^{\Sigma}_{\mathcal{F}})_{\mathrm{W}}, \mathbb{G}_m)$ is bijective.

Finally, as the action of R on each U_n factors through R_n, the limit $\varprojlim_{\mathrm{N}_n}(\mathbb{Z}_p \otimes_{\mathbb{Z}} U_n)$ vanishes and so the argument of Lemma 4.7 implies $\varprojlim^1_{\mathrm{N}_n} U_n$ is isomorphic to $(\mathbb{Z}_p/\mathbb{Z}) \otimes_{\mathbb{Z}} U_a$ for all $a \geq m$. It follows that $\varprojlim^1_{\mathrm{N}_n} U_n$ vanishes if and only if U_a vanishes for all sufficiently large a, or equivalently (by (29)) that B^0_a vanishes for all sufficiently large a. As already observed in the argument of Lemma 6.9, the latter condition is equivalent to requiring that $|\Sigma| = 1$ and the unique place in Σ has full decomposition group in $G_{\mathcal{F}/k}$. This completes the proof of Corollary 6.6.

6.2.4. *Finite-generation.* Since R is not Noetherian, the R-module $H^{1,\Sigma'}_{\mathrm{Iw}}((C^{\Sigma}_{\mathcal{F}})_{\mathrm{W}}, \mathbb{G}_m)/X$ can be finitely-generated even if its submodule $A(\mathcal{F})/X$ is not. However, since $B^0(\mathcal{F})$ is finitely-generated over R (as every place in Σ has an open decomposition group in $G_{\mathcal{F}/k}$), the second displayed exact sequence in Lemma 6.3 implies $H^{1,\Sigma'}_{\mathrm{Iw}}((C^{\Sigma}_{\mathcal{F}})_{\mathrm{W}}, \mathbb{G}_m)/X$ is finitely-generated whenever $A(\mathcal{F})/X$ is.

To analyse the latter condition, we write $A^\ell(\mathcal{F})$ for each prime ℓ for the maximal pro-ℓ subgroup $\varprojlim_{\mathrm{N}_n} A_n[\ell^\infty]$ of $A(\mathcal{F})$. We then note that, since each X_n is finite and or order prime-to-p, the R-module $A^p(\mathcal{F})$ is a direct summand of $A(\mathcal{F})/X$, In particular, if $A(\mathcal{F})/X$ is finitely-generated over R, then so is $A^p(\mathcal{F})$ and so Remark 5.2 implies the characteristic polynomial $\mathrm{char}_\Lambda(A^p(\mathcal{F}))$ is not divisible by $\Phi_i(\gamma)$ for any $i \in \mathbb{N}_0$. We shall therefore henceforth assume this condition to be satisfied.

Next we note that for $w \in \Sigma_{\mathcal{F}_n}$ the complex $\mathsf{R}\Gamma(\kappa(w)_{\mathrm{W}}, \mathbb{G}_m)$ is canonically isomorphic to $\kappa(w)^\times[0]$ and so the long exact sequence of cohomology of (24) gives an exact sequence

$$\bigoplus\nolimits_{w \in \Sigma'(n)} \kappa(w)^\times \to A^{\Sigma'}_{\Sigma}(\mathcal{F}_n) \to A_\Sigma(\mathcal{F}_n) \to 0,$$

in which we set $\Sigma'(n) := \Sigma'_{\mathcal{F}_n}$, Now, as n increases, these sequences are compatible with respect to maps induced by the norms N_n. In addition, since the first term in the sequence is finite, exactness is preserved when passing to the limit over n and so, setting $T_{\Sigma'}(\mathcal{F}) := \varprojlim_{\mathrm{N}_n}\big(\bigoplus_{w \in \Sigma'(n)} \kappa(w)^\times\big)$, one obtains an exact sequence of R-modules of the form

$$T_{\Sigma'}(\mathcal{F}) \to A(\mathcal{F}) \to A_\Sigma(\mathcal{F}) \to 0. \tag{32}$$

Lemma 6.10. $\mu_R\big(T_{\Sigma'}(\mathcal{F})\big) = |\Sigma'|$.

Proof. Since the R-module $T_{\Sigma'}(\mathcal{F})$ decomposes as a direct sum $\bigoplus_{v \in \Sigma'} T_{\{v\}}(\mathcal{F})$, it is enough to consider the case that $|\Sigma'| = 1$. In this case, we fix a place $v' \in \Sigma'_F$ and write $T(v')$ for the limit module $\varprojlim_{\mathrm{N}_n}\big(\bigoplus_{w \in \{v'\}_{\mathcal{F}_n}} \kappa(w)^\times\big)$, so that $T_{\Sigma'}(\mathcal{F}) = \mathbb{Z}[G_{F/k}] \otimes_{\mathbb{Z}[D]} T(v')$, with D the decomposition subgroup of v' in $G_{F/k}$. For each $n \in \mathbb{N}$, we now also set $\Delta_n := G_{\mathcal{F}_n}/F$.

We assume, firstly, that v' splits completely in $\mathcal{F}$ so that

$$T(v') = \varprojlim_n(\mathbb{Z}[\Delta_n] \otimes_{\mathbb{Z}} \kappa(v')^\times) = \mathbb{Z}[\![G_{\mathcal{F}/F}]\!] \otimes_{\mathbb{Z}} \kappa(v')^\times,$$

where the limit is with respect to the maps induced by the natural projections $\mathbb{Z}[\Delta_n] \to \mathbb{Z}[\Delta_{n-1}]$. In particular, if x is any generator of the (cyclic) group $\kappa(v')^\times$, then $T(v')$ is generated over $\mathbb{Z}[\![G_{\mathcal{F}/F}]\!]$

by $1 \otimes x$ and so $\mu_R(T_{\Sigma'}(\mathcal{F})) = 1$, as required. We may thus assume the decomposition subgroup D' of v' in $G_{\mathcal{F}/F}$ is open and hence that $\mathcal{F}^{D'} = \mathcal{F}_{n_0}$ for some $n_0 \in \mathbb{N}$. We fix $w \in \{v'\}_{\mathcal{F}_{n_0}}$ and for $n \geq n_0$ write w_n for the unique place of $\mathcal{F}_n$ above w. Then

$$T(v') = R \otimes_{\mathbb{Z}[\![D']\!]} \varprojlim_{n \geq n_0} \kappa(w_n)^\times,$$

with the limit taken with respect to the maps $\mathrm{N}_n : \kappa(w_n)^\times \to \kappa(w_{n-1})^\times$. To prove $\mu_R(T_{\Sigma'}(\mathcal{F})) = 1$ it is therefore enough to show $\varprojlim_{n \geq n_0} \kappa(w_n)^\times$ is a cyclic $\mathbb{Z}[\![D']\!]$-module. This is clear since for $n > n_0$, the map N_n acts by raising each element of $\kappa(w_n)^\times$ to its $(|\kappa(w_n)^\times|/|\kappa(w_{n-1})^\times|)$-th power and this implies that any generating element of the (cyclic) group $\kappa(w_{n-1})^\times$ has a pre-image under N_n that is a generator of $\kappa(w_n)^\times$. □

For each $b \in \mathbb{N}$, we define a set of rational primes

$$V_b(\mathcal{F}) := \{\ell \in \mathcal{P} \setminus \{p\} : \exists\, m \in \mathbb{N}\,,\, \mathrm{r}(A^\ell_\Sigma(\mathcal{K}_m)) > (b/t_m(\ell))p^m\,\}.$$

We then assume that, for some fixed b and every $n \in \mathbb{N}_0$, one has

$$\bigoplus\nolimits_{\ell \in V_b(\mathcal{F})} (\mathbb{Z}_\ell \otimes_{\mathbb{Z}} A^{\Sigma'}_\Sigma(\mathcal{F}_n)) \subseteq X_n,$$

and we write $\overline{X}$ for the image of X in $A_\Sigma(\mathcal{F})$ under the map in (32). Then the argument of Theorem 5.1(v) implies that $A_\Sigma(\mathcal{F})/\overline{X}$ is finitely-generated over R and hence, via Lemma 6.10 and the exact sequence (32), that $A_\Sigma(\mathcal{F})/X$ is also finitely-generated over R.

In this regard, we also note that, since the number of automorphisms of a finite abelian ℓ-group grows exponentially with its ℓ-rank, Cohen-Lenstra-type heuristics make it seem likely that, at least for large b, the set $V_b(\mathcal{F})$ is comparatively small.

For example, if $\mathcal{F} = F^{\mathrm{con}}$, then the proof of Theorem 5.1(vi) implies that $V_b(F^{\mathrm{con}}) = \emptyset$ for each $b \geq 2g(C_k) + 1$ and so the above argument can be applied with $X = (0)$. We discuss this case in detail in the next section.

6.3. **Constant extensions.** We now fix data $(F/k, \Sigma, \Sigma')$ satisfying Hypothesis 6.1 with respect to the field $\mathcal{K} = k^{\mathrm{con}}$. We also note that, since no place of k either ramifies or splits completely in k^{con}, Hypothesis 6.1(ii) simplifies in this case to the condition that Σ is finite, non-empty and contains all places that ramify in F.

We set $G := G_{F/k}$, $\mathbb{A} := R[G]$ and $\mathbb{A}_n := R_n[G]$ for each n.

6.3.1. *Statement of the main results.* Since the R-module $A_\Sigma(F^{\mathrm{con}})$ is finitely-generated (by Theorem 5.1(vi)), Theorem 6.5 applies with $X = (0)$ to show both that the $\mathbb{A}$-module

$$M^{\Sigma'}_\Sigma(F) := H^{1,\Sigma'}_{\mathrm{dIw}}((C^\Sigma_{F^{\mathrm{con}}})_{\mathrm{W}}, \mathbb{G}_m)$$

is finitely-presented and torsion, and that there exist canonical short exact sequences of $\mathbb{A}$-modules

$$0 \to \mathrm{Fit}^0_{\mathbb{A}}(M^{\Sigma'}_\Sigma(F)) \xrightarrow{\subseteq} \varprojlim_n (\theta_n \cdot \mathbb{A}_n) \xrightarrow{\xi} Z \to 0 \tag{33}$$

and

$$0 \to \theta^{\Sigma'}_{F^{\mathrm{con}},\Sigma} \cdot \mathbb{A} \xrightarrow{\subseteq} \varprojlim_n (\theta_n \cdot \mathbb{A}_n) \xrightarrow{\xi'} Z \to 0, \tag{34}$$

in which Z denotes the derived limit $\varprojlim^1_n \mathrm{Ann}_{\mathbb{A}_n}(\theta_n)$.

In particular, since the map ξ arises as a canonical connecting homomorphism in the theory of pro-discrete modules, the sequence (33) completely determines the initial Fitting ideal of $M^{\Sigma'}_\Sigma(F)$ over the ring $\mathbb{A}$ (which is neither coherent, compact nor regular and has infinite Krull dimension) in terms of Stickelberger elements. In the next two results we show that, under certain additional conditions, the sequences (33) and (34) can also be combined to give a more explicit link of this nature.

The following result will be proved in §6.3.2.

Theorem 6.11. *Assume there exists no non-trivial unramified extension of k in F and, in addition, that no place ramifying in F is of degree divisible by p. Then the following claims are valid.*

(i) *Set $\mathcal{I} := \mathrm{Fit}^0_{\mathbb{A}}(M^{\Sigma'}_{\Sigma}(F)) \cap (\theta^{\Sigma'}_{F^{\mathrm{con}},\Sigma}\cdot\mathbb{A})$. Then both $\mathrm{Fit}^0_{\mathbb{A}}(M^{\Sigma'}_{\Sigma}(F))/\mathcal{I}$ and $(\theta^{\Sigma'}_{F^{\mathrm{con}},\Sigma}\cdot\mathbb{A})/\mathcal{I}$ are finite and have order that is prime-to-p and divisible only by primes that divide $|G|$.*

(ii) *If either $|G|$ is a power of p or F/k ramifies at a single place, then*

$$\mathrm{Fit}^0_{\mathbb{A}}\big(M^{\Sigma'}_{\Sigma}(F)\big) = \theta^{\Sigma'}_{F^{\mathrm{con}},\Sigma}\cdot\mathbb{A}.$$

Remark 6.12. Let k_1 be any function field, fix a place v_∞ of k_1 and write A for the subring of k_1 comprising elements that are regular at every place other than v_∞. Fix a finite (possibly empty) set Σ_1 of prime ideals of A whose respective degrees are not divisible by p and write $\mathfrak{m}$ for the product of ideals in Σ_1. Then the real (relative to v_∞) ray class field $H(\mathfrak{m})$ of conductor $\mathfrak{m}$ is a finite abelian extension of k_1 in which v_∞ splits completely and all places that ramify belong to Σ_1. In particular, if we set $k := H(A)$, then the field $H(\mathfrak{m}^\infty) := \bigcup_{t\in\mathbb{N}} H(\mathfrak{m}^t)$ is an abelian extension of k that ramifies at every place in $\Sigma_{1,k}$ (and at no other places), contains no unramified extension of k and is such that $G_{H(\mathfrak{m}^\infty)/k}$ has an open subgroup (and hence also a quotient) that is isomorphic to $\mathbb{Z}_p^{\mathbb{N}}$, thereby giving an abundant supply of finite extensions F/k satisfying the hypotheses of Theorem 6.11(ii). We further note that the extensions $H(\mathfrak{m}^\infty)/k_1$ incorporate the Drinfeld modular towers considered by Bley and Popescu in [4] and hence also the Carlitz-Hayes cyclotomic extensions considered by Anglès et al in [1].

In any situation in which the equality in Theorem 6.11(ii) is valid, the $\mathbb{A}$-module $\mathrm{Fit}^0_{\mathbb{A}}\big(M^{\Sigma'}_{\Sigma}(F)\big)$ is free of rank 1. If p is not exceptional (but no hypotheses are imposed on the ramification of F/k) it is possible to interpret this condition on $\mathrm{Fit}^0_{\mathbb{A}}\big(M^{\Sigma'}_{\Sigma}(F)\big)$ in terms of a natural Euler characteristic of the complex $\mathsf{R}\Gamma^{\Sigma'}((C^{\Sigma}_{F^{\mathrm{con}}})_{\mathrm{W}},\mathbb{G}_m)$. To state this result, we write $\mathfrak{M}^{\mathrm{lc}}_G$ for the category of finitely-generated $\mathbb{A}$-modules that are locally-free (and hence projective) of constant rank. We denote the Grothendieck group of $\mathfrak{M}^{\mathrm{lc}}_G$ by $K_0^{\mathrm{lc}}(\mathbb{A})$ and use the canonical 'determinant' homomorphism $\delta^{\mathrm{lc}}_{\mathbb{A}} : K_0^{\mathrm{lc}}(\mathbb{A}) \to \mathrm{Pic}(\mathbb{A})$. We shall also say that an object X of $\mathsf{D}(\mathbb{A})$ is 'strictly-perfect' if it is isomorphic (in the derived category) to a bounded complex of modules in $\mathfrak{M}^{\mathrm{lc}}_G$ and we note that each such X has a well-defined 'Euler characteristic' $\chi^{\mathrm{lc}}_{\mathbb{A}}(X)$ in $K_0^{\mathrm{lc}}(\mathbb{A})$ (that depends only on X up to isomorphism in $\mathsf{D}(\mathbb{A})$). Finally, we write Δ_G for the natural (diagonal) projection map $\mathrm{Pic}(\mathbb{A}) \to \varprojlim_n \mathrm{Pic}(\mathbb{A}_n)$.

The following result will be proved in §6.3.3.

Proposition 6.13. *Set $C^{\Sigma'}_{\Sigma}(F) := \mathsf{R}\Gamma^{\Sigma'}((C^{\Sigma}_{F^{\mathrm{con}}})_{\mathrm{W}},\mathbb{G}_m)$ and $\mathcal{I}^{\Sigma'}_{\Sigma}(F) := \mathrm{Fit}^0_{\mathbb{A}}(M^{\Sigma'}_{\Sigma}(F))$. Then, if p is not exceptional, the following claims are valid.*

(i) *$C^{\Sigma'}_{\Sigma}(F)$ is strictly-perfect as a complex of $\mathbb{A}$-modules.*

(ii) *$\mathcal{I}^{\Sigma'}_{\Sigma}(F)$ is an invertible ideal of $\mathbb{A}$ and its class in $\mathrm{Pic}(\mathbb{A})$ is both equal to $\delta^{\mathrm{lc}}_{\mathbb{A}}\big(\chi^{\mathrm{lc}}_{\mathbb{A}}(C^{\Sigma'}_{\Sigma}(F))\big)$ and belongs to $\ker(\Delta_G)$.*

(iii) *$\bigwedge^{|G|}_R(\mathcal{I}^{\Sigma'}_{\Sigma}(F))$ is a free R-module of rank 1.*

6.3.2. *The proof of Theorem 6.11.* We first establish a useful reduction result.

Lemma 6.14. *Let v be a place of k outside $\Sigma\cup\Sigma'$ and, since v is unramified in F^{con}, write $\sigma_v = \sigma_v(F)$ for its Frobenius automorphism in $\mathrm{Gal}(F^{\mathrm{con}}/k)$. Then $1-\sigma_v$ is a non-zero divisor in $\mathbb{A}$ and there are equalities*

$$\theta^{\Sigma'}_{F^{\mathrm{con}},\Sigma\cup\{v\}} = (1-\sigma_v)\theta^{\Sigma'}_{F^{\mathrm{con}},\Sigma} \quad \textit{and} \quad \mathrm{Fit}^0_{\mathbb{A}}(M^{\Sigma'}_{\Sigma\cup\{v\}}(F)) = (1-\sigma_v)\mathrm{Fit}^0_{\mathbb{A}}(M^{\Sigma'}_{\Sigma}(F)).$$

Proof. To show that $1-\sigma_v$ is a non-zero divisor in $\mathbb{A}$, it is enough to note $\sigma_v^{|G|}$ generates an open subgroup of $G_{F^{\mathrm{con}}/F}$. Indeed, this implies that the product $(1-\sigma_v)\sum_{i=0}^{i=|G|-1}\sigma_v^i = 1-\sigma_v^{|G|}$ is a non-zero divisor in $\mathbb{Z}_p[\![G_{F^{\rm con}/F}]\!] \cong \mathbb{Z}_p[\![\mathbb{Z}_p]\!]$, and hence also in $\mathbb{A}$. In addition, the first displayed equality expresses a well-known property of truncated Stickelberger elements.

It is therefore enough to prove the second displayed equality, and for this we set $\Sigma_1 := \Sigma \cup \{v\}$. Then, for each n, there exists a canonical exact triangle in $\mathsf{D}^{\rm perf}(\mathbb{A}_n)$ of the form

$$\mathsf{R}\Gamma^{\Sigma'}((C^{\Sigma}_{F_n})_{\rm W}, \mathbb{G}_m) \to \mathsf{R}\Gamma^{\Sigma'}((C^{\Sigma_1}_{F_n})_{\rm W}, \mathbb{G}_m) \to \mathsf{R}\mathrm{Hom}_{\mathbb{A}_n}(\mathsf{R}\Gamma(\kappa(v)_{\rm W}, \mathbb{A}_n), \mathbb{A}_n[-1]) \to \cdot$$

that is compatible with the natural projection morphisms as n varies (cf. [8, Prop. 2.4(ii) and Rem. 2.5(i)]). In addition, for each $n > 1$, the third complex in this triangle is canonically isomorphic to the complex $\mathbb{A}_n \to \mathbb{A}_n$ in which the first term occurs in degree 0 and the differential sends each x to $(1-\sigma_v)x$, and the natural projection isomorphism

$$\mathbb{A}_{n-1} \otimes^{\mathbb{L}}_{\mathbb{A}_n} \mathsf{R}\mathrm{Hom}_{\mathbb{A}_n}(\mathsf{R}\Gamma(\kappa(v)_{\rm W}, \mathbb{A}_n), \mathbb{A}_n[-1]) \cong \mathrm{Hom}_{\mathbb{A}_{n-1}}(\mathsf{R}\Gamma(\kappa(v)_{\rm W}, \mathbb{A}_{n-1}), \mathbb{A}_{n-1}[-1])$$

is induced by the morphism of complexes of $\mathbb{A}_n$-modules

$$\begin{array}{ccc} \mathbb{A}_n & \xrightarrow{x \mapsto (1-\sigma_v)x} & \mathbb{A}_n \\ \big\downarrow & & \big\downarrow \\ \mathbb{A}_{n-1} & \xrightarrow{x \mapsto (1-\sigma_v)x} & \mathbb{A}_{n-1} \end{array}$$

in which both vertical maps are the canonical projection. Hence, upon passing to the inverse limit over n of the above triangles, one deduces from Proposition 4.11(ii) (and the argument given at the end of the proof of Proposition 4.11(iv)), the existence of an exact triangle in $\mathsf{D}(\mathbb{A})$ of the form

$$\mathsf{R}\Gamma^{\Sigma'}((C^{\Sigma}_{F^{\rm con}})_{\rm W}, \mathbb{G}_m) \to \mathsf{R}\Gamma^{\Sigma'}((C^{\Sigma_1}_{F^{\rm con}})_{\rm W}, \mathbb{G}_m) \to \big[\mathbb{A} \xrightarrow{\phi_v} \mathbb{A}\big] \to \cdot$$

in which the first module in the third complex occurs in degree 0 and ϕ_v sends x to $(1-\sigma_v)x$. In particular, since ϕ_v is injective (as $1-\sigma_v$ is a non-zero divisor), the long exact cohomology sequence of this triangle combines with Lemma 6.4 with $X = (0)$ to give an exact sequence of $\mathbb{A}$-modules $0 \to M^{\Sigma'}_{\Sigma}(F) \to M^{\Sigma'}_{\Sigma_1}(F) \to \mathrm{cok}(\phi_v) \to 0$. Hence, since the first two modules in this sequence are finitely-presented (by Lemma 6.4 and Theorem 6.5 with $X = (0)$) and the third lies in a short exact sequence of the form

$$0 \to \mathbb{A} \xrightarrow{\phi_v} \mathbb{A} \to \mathrm{cok}(\phi_v) \to 0, \tag{35}$$

a general algebraic result of Cornacchia and Greither [11, Lem. 3] implies that

$$\mathrm{Fit}^0_{\mathbb{A}}(M^{\Sigma'}_{\Sigma_1}(F)) = \mathrm{Fit}^0_{\mathbb{A}}(\mathrm{cok}(\phi_v))\mathrm{Fit}^0_{\mathbb{A}}(M^{\Sigma'}_{\Sigma}(F)).$$

It is therefore now enough to note that the short exact sequence (35) combines with the explicit definition of Fitting ideals to imply $\mathrm{Fit}^0_{\mathbb{A}}(\mathrm{cok}(\phi_v))$ is the $\mathbb{A}$-ideal generated by $\det(\phi_v) = 1 - \sigma_v$. □

Taking account of Lemma 6.14, the proof of Theorem 6.11 is reduced to the case that Σ is the set of places that ramify in F. It is then convenient to first treat the case that $\Sigma = \{v\}$ comprises a single place. In this case, v is both totally ramified in F (since F/k is assumed to have no non-trivial unramified subextension) and inert in $k^{\rm con}$ (since its degree is assumed to be prime-to-p) and so has full decomposition group in $G_{F^{\rm con}/k}$. Hence, by the final assertion of Theorem 6.5, the derived limit Z that occurs in the exact sequences (33) and (34) vanishes. In this case, therefore, the latter sequences combine directly to imply that

$$\mathrm{Fit}^0_{\mathbb{A}}(M^{\Sigma'}_{\Sigma}(F)) = \varprojlim_n (\theta_n \cdot \mathbb{A}_n) = \theta^{\Sigma'}_{F^{\rm con},\Sigma} \cdot \mathbb{A}. \tag{36}$$

We note, in particular, that this equality verifies the second assertion of Theorem 6.11(ii) and that the remainder of (ii) then follows directly from Theorem 6.11(i).

It is therefore enough for us to prove Theorem 6.11(i) and to do this we set $m := |G|$, $\mathbb{A} := \mathbb{A}[1/m]$ and $\mathbb{A}'_n := \mathbb{A}_n[1/m]$ for each $n \in \mathbb{N}_0$. We then claim it suffices to prove that

$$\mathrm{Fit}^0_{R[G]}\big(M^{\Sigma'}_{\Sigma}(F))\big)[1/m] = \theta^{\Sigma'}_{F^{\mathrm{con}},\Sigma} \cdot \mathbb{A}'. \tag{37}$$

Indeed, if this is true, then, since the $\mathbb{A}$-module $M^{\Sigma'}_{\Sigma}(F)$ is finitely-presented (by Theorem 6.5), and hence the $\mathbb{A}$-module $\mathrm{Fit}^0_{\mathbb{A}}(M^{\Sigma'}_{\Sigma}(F))$ is finitely-generated, for any sufficiently large integer t one has

$$m^t(\theta^{\Sigma'}_{F^{\mathrm{con}},\Sigma} \cdot \mathbb{A}) \subseteq \mathrm{Fit}^0_{\mathbb{A}}(M^{\Sigma'}_{\Sigma}(F)) \subseteq m^{-t}(\theta^{\Sigma'}_{F^{\mathrm{con}},\Sigma} \cdot \mathbb{A}). \tag{38}$$

In particular, the first inclusion here combines with the exact sequences (33) and (34) to imply ξ sends $\theta^{\Sigma'}_{F^{\mathrm{con}},\Sigma} \cdot \mathbb{A}$ into the submodule $Z[m^t]$ of Z comprising elements that are annihilated by m^t. However, from the proof of Lemma 6.9(ii), one knows that Z is equal to $(\mathbb{Z}_p/\mathbb{Z}) \otimes_{\mathbb{Z}} Y$ for a (finitely-generated) lattice Y and so $Z[m^t]$ is both finite and of order prime-to-p. The quotient module

$$(\theta^{\Sigma'}_{F^{\mathrm{con}},\Sigma} \cdot \mathbb{A})/\mathcal{I} \cong \big(\mathrm{Fit}^0_{\mathbb{A}}(M^{\Sigma'}_{\Sigma}(F)) + \theta^{\Sigma'}_{F^{\mathrm{con}},\Sigma} \cdot \mathbb{A}\big)/\mathrm{Fit}^0_{\mathbb{A}}(M^{\Sigma'}_{\Sigma}(F)) \xhookrightarrow{\xi} Z[m^t]$$

is therefore also finite and of order prime to p, as required. In addition, the analogous result for the quotient $\mathrm{Fit}^0_{\mathbb{A}}(M^{\Sigma'}_{\Sigma}(F))/\mathcal{I}$ is derived from the second inclusion of (38) in an entirely similar way.

To complete the proof of Theorem 6.11, we are thus reduced to proving (37) and, to do this, we shall argue by induction on $|\Sigma|$. Since the case $|\Sigma| = 1$ follows immediately from (36), we therefore assume $|\Sigma| > 1$. We then write $\mathfrak{S} = \mathfrak{S}_{F/k}$ for the set of intermediate fields of F/k and, for $E \in \mathfrak{S}$, we denote the subset of Σ comprising places that ramify in E by $\Sigma(E)$ (so $\Sigma(F) = \Sigma$). For $\psi \in G^*$, we also write $F_\psi \in \mathfrak{S}$ for the fixed field of F under $\ker(\psi)$. Then, for each $E \in \mathfrak{S}$, the set

$$\Xi(E) := \{\psi \in G^* : F_\psi = E\}$$

is a (possibly empty) conjugacy class for the action of $G_{\mathbb{Q}^c/\mathbb{Q}}$ on G^* and so the idempotent

$$\varepsilon_E := \sum\nolimits_{\psi \in \Xi(E)} e_\psi$$

belongs to $\mathbb{Z}[1/m][G]$ and, in $\mathbb{Q}^c[G]$, one has

$$1 = \sum\nolimits_{\psi \in G^*} e_\psi = \sum\nolimits_{E \in \mathfrak{S}} \big(\sum\nolimits_{\psi \in \Xi(E)} e_\psi\big) = \sum\nolimits_{E \in \mathfrak{S}} \varepsilon_E = \sum\nolimits_{E \in \mathfrak{S}_1} \varepsilon_E + \sum\nolimits_{E \in \mathfrak{S}_2} \varepsilon_E, \tag{39}$$

with $\mathfrak{S}_1 := \{E \in \mathfrak{S} : \Sigma(E) = \Sigma\}$ and $\mathfrak{S}_2 := \{E \in \mathfrak{S} : \Sigma(E) \neq \Sigma\}$.

Take $E \in \mathfrak{S}_1$ and, for each $n \in \mathbb{N}_0$, identify ε_E as an idempotent of the ring $R_n[G_E][1/m] = \mathbb{Z}[1/m][G_{E_n/k}]$. Then, since $\Sigma(E) = \Sigma$, one has $\varepsilon_E = \sum_\psi e_\psi$, where ψ runs over a set of characters in $G^*_{E_n/k}$ that cannot be trivial on the decomposition group of any place in Σ. In particular, from the equivalence (31) (and the sentence that follows it) one knows ε_E annihilates $\mathrm{Ann}_{R_n[G]}(\theta_n)[1/m]$. This in turn implies ε_E annihilates $Z[1/m]$ and so the (scalar extensions of the) exact sequences (33) and (34) combine to imply that

$$\varepsilon_E\big(\mathrm{Fit}^0_{\mathbb{A}}(M^{\Sigma'}_{\Sigma}(F))[1/m]\big) = \varepsilon_E\big((\varprojlim\nolimits_n(\theta_n \cdot \mathbb{A}_n))[1/m]\big) = \varepsilon_E\big(\theta^{\Sigma'}_{F^{\mathrm{con}},\Sigma} \cdot \mathbb{A}'\big). \tag{40}$$

We now take $E \in \mathfrak{S}_2$ and set $G^E := G_{F/E}$, $G_E := G_{E/k}$, $\mathbb{A}_E := R[G_E]$ and $\mathbb{A}'_E := R'[G_E]$. We also write e_E for the idempotent $(1/|G^E|)\sum_{h \in G^E} h$ of $\mathbb{A}'$ and π_E for the projection $\mathbb{A}' \to \mathbb{A}'_E$. Then, since $\mathbb{A}'_E$ identifies with the direct summand $e_E\mathbb{A}'$ of $\mathbb{A}'$ the functor $N \mapsto \mathbb{A}'_E \otimes_{\mathbb{A}'} N$ is exact on the category of $\mathbb{A}'$-modules and $\mathbb{A}'_E \otimes_{\mathbb{A}'} N$ identifies with the submodule N^{G^E} of N. In particular, since for each n there exists a canonical isomorphism in $\mathsf{D}^{\mathrm{perf}}(\mathbb{A}_n)$

$$\mathsf{R}\mathrm{Hom}_{G^E}(\mathbb{Z}, \mathsf{R}\Gamma^{\Sigma'}((C^{\Sigma}_{F_n})_{\mathrm{W}}, \mathbb{G}_m)) \cong \mathbb{A}_{E,n} \otimes^{\mathbb{L}}_{\mathbb{A}_n} \mathsf{R}\Gamma^{\Sigma'}((C^{\Sigma}_{F_n})_{\mathrm{W}}, \mathbb{G}_m) \cong \mathsf{R}\Gamma^{\Sigma'}((C^{\Sigma}_{E_n})_{\mathrm{W}}, \mathbb{G}_m)$$

(cf. [6, Lem. 1(iv)]), there exists a composite isomorphism in $\mathsf{D}(\mathbb{A}'_E)$ of the form

$$\begin{aligned}\mathbb{A}'_E \otimes^{\mathbb{L}}_{\mathbb{A}} \mathsf{R}\Gamma^{\Sigma'}((C^{\Sigma}_{F^{\rm con}})_{\rm W}, \mathbb{G}_m) &\cong \big(\mathrm{R}\varprojlim_{\theta_n} \mathsf{R}\mathrm{Hom}_{G^E}(\mathbb{Z}, \mathsf{R}\Gamma^{\Sigma'}((C^{\Sigma}_{F_n})_{\rm W}, \mathbb{G}_m))\big)[1/m]\\ &\cong \mathrm{R}\varprojlim_{\theta_n} \big(\mathsf{R}\Gamma^{\Sigma'}((C^{\Sigma}_{E_n})_{\rm W}, \mathbb{G}_m)\big)[1/m]\\ &\cong \mathsf{R}\Gamma^{\Sigma'}((C^{\Sigma}_{E^{\rm con}})_{\rm W}, \mathbb{G}_m)[1/m].\end{aligned}$$

Since $\mathsf{R}\Gamma^{\Sigma'}((C^{\Sigma}_{F^{\rm con}})_{\rm W}, \mathbb{G}_m)$ is acyclic in degrees greater than 1 (by Lemma 6.4 with $X = (0)$), this isomorphism in turn induces an isomorphism of $\mathbb{A}'_E$-modules

$$\begin{aligned}\mathbb{A}'_E \otimes_{\mathbb{A}'} (M^{\Sigma'}_{\Sigma}(F)[1/m]) &= (\mathbb{A}'_E \otimes_{\mathbb{A}'} H^1(\mathsf{R}\Gamma^{\Sigma'}((C^{\Sigma}_{F^{\rm con}})_{\rm W}, \mathbb{G}_m)))[1/m]\\ &\cong H^1(\mathbb{A}'_E \otimes^{\mathbb{L}}_{\mathbb{A}} \mathsf{R}\Gamma^{\Sigma'}((C^{\Sigma}_{F^{\rm con}})_{\rm W}, \mathbb{G}_m))[1/m]\\ &\cong H^1(\mathsf{R}\Gamma^{\Sigma'}((C^{\Sigma}_{E^{\rm con}})_{\rm W}, \mathbb{G}_m))[1/m] = M^{\Sigma'}_{\Sigma}(E)[1/m].\end{aligned} \tag{41}$$

Noting $\varepsilon_E = \varepsilon_E e_E$, we can therefore now compute in $\varepsilon_E\mathbb{A}' = \varepsilon_E(e_E\mathbb{A}') = \varepsilon_E\mathbb{A}'_E$ that

$$\begin{aligned}\varepsilon_E(\mathrm{Fit}^0_{\mathbb{A}}(M^{\Sigma'}_{\Sigma}(F))[1/m]) &= \varepsilon_E \cdot \pi_E(\mathrm{Fit}^0_{\mathbb{A}'}(M^{\Sigma'}_{\Sigma}(F)[1/m]))\\ &= \varepsilon_E \cdot \mathrm{Fit}^0_{\mathbb{A}'_E}(\mathbb{A}'_E \otimes_{\mathbb{A}'} M^{\Sigma'}_{\Sigma}(F)[1/m])\\ &= \varepsilon_E \cdot \mathrm{Fit}^0_{\mathbb{A}'_E}(M^{\Sigma'}_{\Sigma}(E)[1/m])\\ &= \varepsilon_E\big(\textstyle\prod_{v\in\Sigma\setminus\Sigma(E)}(1-\sigma_v)\big) \cdot \mathrm{Fit}^0_{\mathbb{A}_E}\big(M^{\Sigma'}_{\Sigma(E)}(E)\big)\big)[1/m]\\ &= \varepsilon_E\big(\textstyle\prod_{v\in\Sigma\setminus\Sigma(E)}(1-\sigma_v)\big)\theta^{\Sigma'}_{E^{\rm con},\Sigma(E)} \cdot \mathbb{A}'_E\\ &= \varepsilon_E(\theta^{\Sigma'}_{F^{\rm con},\Sigma} \cdot \mathbb{A}').\end{aligned}$$

Here the second equality follows from a standard property of Fitting ideals under ring projection, the third from the isomorphism (41), the fourth from Lemma 6.14, the fifth from the inductive hypothesis (since $E \in \mathfrak{S}_2$ implies $|\Sigma(E)| < |\Sigma|$) and the last from Lemma 6.14 and the equality $\pi_E(\theta^{\Sigma'}_{F^{\rm con},\Sigma}) = \theta^{\Sigma'}_{E^{\rm con},\Sigma}$.

The claimed equality (37) now follows upon combining the last displayed equality (for each $E \in \mathfrak{S}_2$) with (40) (for each $E \in \mathfrak{S}_1$) and the decomposition (39) of 1 as a sum of the mutually orthogonal idempotents ε_E. This completes the proof of Theorem 6.11.

6.3.3. *The proof of Proposition 6.13.* In this argument we set $C^{\bullet}_F := \mathsf{R}\Gamma^{\Sigma'}((C^{\Sigma}_{F^{\rm con}})_{\rm W}, \mathbb{G}_m)$ and assume p is not exceptional.

In this case, the observations made at the beginning of §6.2.2 (with $X = (0)$ and $\mathcal{F} = F^{\rm con}$) combine with Proposition 4.11(iv) to imply the existence of an isomorphism in $\mathsf{D}(\mathbb{A})$

$$C^{\bullet}_F \cong \big[P \xrightarrow{\theta} \mathbb{A}^d\big], \tag{42}$$

in which P is a finitely-generated projective $\mathbb{A}$-module placed in degree 0 and d is a natural number. From Lemma 6.4 it then follows that $\ker(\theta)$ and $\mathrm{cok}(\theta)$ are respectively isomorphic to $\varprojlim_{\mathrm{N}_n} \mathcal{O}^{\times,\Sigma'}_{F_n,\Sigma}$ and $M^{\Sigma'}_{\Sigma}(F)$. In particular, since the isomorphisms (29) combine with the observations made just after (31) to imply that $\mathcal{O}^{\times,\Sigma'}_{F_n,\Sigma} = \mathcal{O}^{\times,\Sigma'}_{F_{n+1},\Sigma}$ for all sufficiently large n, the module $\ker(\theta)$ vanishes. Then, since Theorem 6.5 implies the $\mathbb{A}$-module $\mathrm{cok}(\theta)$ is torsion, the isomorphism (42) implies that the $\mathbb{A}$-module P is locally-free of constant rank d and hence that $C^{\bullet}_F$ is a strictly-perfect complex of $\mathbb{A}$-modules, thereby proving claim (i).

We now fix a natural number $d' \geq d$ and a surjective morphism of $\mathbb{A}$-modules $\pi : \mathbb{A}^{d'} \to P$, and consider the commutative diagram

$$\begin{tikzcd}
\bigwedge_{\mathbb{A}}^{d}\mathbb{A}^{d'} \arrow[d, "\bigwedge_{\mathbb{A}}^{d}\pi"'] \arrow[dr, "\bigwedge_{\mathbb{A}}^{d}(\theta\circ\pi)"] & \\
\bigwedge_{\mathbb{A}}^{d}P \arrow[r, "\bigwedge_{\mathbb{A}}^{d}\theta"] & \bigwedge_{\mathbb{A}}^{d}\mathbb{A}^{d} \cong \mathbb{A}.
\end{tikzcd}$$

Then, since $\theta\circ\pi$ defines a free presentation of $M_{\Sigma}^{\Sigma'}(F)$, if one computes $\mathrm{im}(\bigwedge_{\mathbb{A}}^{d}(\theta\circ\pi))$ in terms of the matrix representing $\theta\circ\pi$ (with respect to the standard bases of $\mathbb{A}^{d'}$ and $\mathbb{A}^{d}$), the explicit definition of Fitting ideal implies that

$$\mathrm{Fit}^0_{\mathbb{A}}(M_{\Sigma}^{\Sigma'}(F)) = \mathrm{im}(\bigwedge\nolimits_{\mathbb{A}}^{d}(\theta\circ\pi)) = \mathrm{im}(\bigwedge\nolimits_{\mathbb{A}}^{d}\theta) \cong \bigwedge\nolimits_{\mathbb{A}}^{d}P.$$

Here the second equality follows from the surjectivity of $\bigwedge_{\mathbb{A}}^{d}\pi$ in the above commutative diagram and the isomorphism (of $\mathbb{A}$-modules) is induced by the fact $\bigwedge_{\mathbb{A}}^{d}\theta$ is injective since $\bigwedge_{\mathbb{A}}^{d}P$ is an invertible $\mathbb{A}$-module. From the resolution (42), we can therefore deduce that $\delta_{\mathbb{A}}^{\mathrm{lc}}\big(\chi_{\mathbb{A}}^{\mathrm{lc}}(C_F^\bullet)\big)$ is the isomorphism class of the invertible $\mathbb{A}$-module

$$\big(\bigwedge\nolimits_{\mathbb{A}}^{d}P\big)\otimes_{\mathbb{A}}\mathrm{Hom}_{\mathbb{A}}\big(\bigwedge\nolimits_{\mathbb{A}}^{d}\mathbb{A}^{d},\mathbb{A}\big) \cong \bigwedge\nolimits_{\mathbb{A}}^{d}P \cong \mathrm{Fit}^0_{\mathbb{A}}(M_{\Sigma}^{\Sigma'}(F)),$$

as required.

To complete the proof of claim (ii), it is now enough to show, for each n, that $\chi_{\mathbb{A}}^{\mathrm{lc}}(C_F^\bullet)$ belongs to the kernel of the projection map $\pi_n' : K_0^{\mathrm{lc}}(\mathbb{A}) \to K_0(\mathbb{A}_n)$. In addition, since $\pi_n'(\chi_{\mathbb{A}}^{\mathrm{lc}}(C_F^\bullet))$ is equal to the Euler characteristic of the complex $\mathbb{A}_n \otimes_{\mathbb{A}}^{\mathbb{L}} C_F^\bullet \cong \mathsf{R}\Gamma^{\Sigma'}((C_{F_n}^{\Sigma})_{\mathrm{W}},\mathbb{G}_m)$ in $\mathsf{D}^{\mathrm{perf}}(\mathbb{A}_n)$, this required vanishing follows from [8, Rem. 3.3(ii)].

To prove claim (iii) the key point is that, if G is the trivial group (so $\mathbb{A} = R$), then results of Kervaire and Murthy [19, Th. 1.3] and Ullom [31, Cor. 2.6] combine to imply that if p is not exceptional, then Δ_G is bijective (cf. Stolin [26, Th. 4.1] and [7, Rem. 3.9]). In particular, to prove (iii) it is enough to show, for each n, that the class of $\bigwedge_R^{|G|}(\mathcal{I}_{\Sigma}^{\Sigma'}(F))$ belongs to the kernel of the projection map $\mathrm{Pic}(R) \to \mathrm{Pic}(R_n)$. Moreover, since the class of $\bigwedge_R^{|G|}(\mathcal{I}_{\Sigma}^{\Sigma'}(F))$ is the image of $\chi_{\mathbb{A}}^{\mathrm{lc}}(C_F^\bullet)$ under the composite morphism $K_0^{\mathrm{lc}}(\mathbb{A}) \to K_0^{\mathrm{lc}}(R) \to \mathrm{Pic}(R)$, in which the first map is induced by restriction of scalars and the second is δ_G for G trivial, the required vanishing follows from the vanishing of $\pi_n'(\chi_{\mathbb{A}}^{\mathrm{lc}}(C_F^\bullet))$ observed above.

This completes the proof of Proposition 6.13.

6.3.4. *An integral main conjecture.* Taken together, the observations and results of this section suggest to us the following 'main conjecture of equivariant integral Iwasawa theory'.

Conjecture 6.15. *If $(F/k,\Sigma,\Sigma')$ is any data satisfying Hypothesis 6.1 with respect to $\mathcal{K} = k^{\mathrm{con}}$, then the (finitely-presented) $\mathbb{Z}[\![\mathbb{Z}_p\times G]\!]$-module $H^{1,\Sigma'}_{\mathrm{dIw}}((C^{\Sigma}_{F^{\mathrm{con}}})_{\mathrm{W}},\mathbb{G}_m)$ is such that*

$$\mathrm{Fit}^0_{\mathbb{Z}[\![\mathbb{Z}_p\times G]\!]}\big(H^{1,\Sigma'}_{\mathrm{dIw}}((C^{\Sigma}_{F^{\mathrm{con}}})_{\mathrm{W}},\mathbb{G}_m)\big) = \theta^{\Sigma'}_{F^{\mathrm{con}},\Sigma}\cdot\mathbb{Z}[\![\mathbb{Z}_p\times G]\!].$$

In particular, for all such data the ideal $\mathrm{Fit}^0_{\mathbb{Z}[\![\mathbb{Z}_p\times G]\!]}\big(H^{1,\Sigma'}_{\mathrm{dIw}}((C^{\Sigma}_{F^{\mathrm{con}}})_{\mathrm{W}},\mathbb{G}_m)\big)$ is principal.

Remark 6.16. The precise link between Stickelberger elements and the Galois structure of the (derived Iwasawa) Weil-étale cohomology of $\mathbb{G}_m$ that is predicted by this conjecture is striking in being formulated relative to a ring $\mathbb{A} = \mathbb{Z}[\![\mathbb{Z}_p\times G]\!]$ that is neither coherent, compact nor regular and has infinite Krull dimension. Nevertheless, the conjecture is fully verified under mild additional hypotheses by Theorem 6.11(ii) and Proposition 6.13 also provides further support for the prediction that $\mathrm{Fit}^0_{\mathbb{A}}\big(H^{1,\Sigma'}_{\mathrm{dIw}}((C^{\Sigma}_{F^{\mathrm{con}}})_{\mathrm{W}},\mathbb{G}_m)\big)$ is principal. In general, the validity of Conjecture 6.15 would follow directly from a comparison of the short exact sequences (33) and (34) if one could show that the canonical connecting homomorphisms ξ and ξ' coincide. (In fact, a closer analysis shows it is enough for the derivation of the predicted equality that one has either $\xi = \lambda\cdot\xi'$ or $\xi' = \lambda\cdot\xi$ for some element λ of $\mathbb{A}$ with the property that, for any fixed integer $n \geq n_0$, with n_0 as in the proof of Lemma 6.9, the quotient module $\mathbb{A}_n/(\theta_n, \varrho_n(\lambda))$ is finite and of p-power order). However, even if G is trivial, the results obtained so far are not enough to establish any such relation. In this direction, for example, Proposition 3.2(iii) shows that there exists an infinite chain $J(1) \supset J(2) \supset \cdots$ of principal ideals of R in which, for every $i \in \mathbb{N}$, the index of $J(i+1)$ in $J(i)$ is non-trivial, finite and prime-to-p (in analogy to Theorem 6.11(i)) and yet, for each $n \in \mathbb{N}$, there exists (in analogy to (33) and (34)) an exact sequence of R-modules $0 \to J(i) \to \varprojlim_n \varrho_n(J(1)) \to \mathbb{Z}_p/\mathbb{Z} \to 0$ in which the third arrow is the respective canonical connecting homomorphism (see (5)) and, in addition, (in analogy to (30)) one has $\varrho_n(J(i)) = \varrho_n(J(1))$.

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