# ON NON-COMMUTATIVE EULER SYSTEMS, II: GENERAL THEORY AND EXAMPLES

*TO THE MEMORY OF JAN NEKOVÁŘ, FRIEND AND COLLEAGUE*

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Abstract. Let $p$ be a prime, $T$ a $p$-adic representation over a number field $K$ and $\mathcal{K}$ an arbitrary Galois extension of $K$. For each non-negative integer $r$, we introduce a notion of 'non-commutative Euler system of rank $r$' for $T$ relative to $\mathcal{K}/K$. We prove that, if $p$ is odd and $T$ and $\mathcal{K}$ satisfy standard hypotheses, then the values of such systems annihilate, as Galois modules, the Selmer groups of $T$ over finite (possibly non-abelian) Galois extensions of $K$ in $\mathcal{K}$. Under mild hypotheses on $T$ and $\mathcal{K}$, we also give an unconditional construction of a canonical family of non-commutative Euler systems of rank dependent on $T$. As a concrete application of this construction, we extend the classical Euler system of cyclotomic units to the setting of arbitrary totally real Galois extensions of $\mathbb{Q}$ and describe explicit links between this extended cyclotomic Euler system, the values at zero of derivatives of Artin $L$-series and the Galois structures of ideal class groups.

## 1. Introduction

Since its introduction by Kolyvagin, Rubin and Thaine in the 1980's, the theory of Euler systems has played a vital role in the proof of results concerning Selmer groups of a $p$-adic representation $T$ over a number field $K$. The theory itself has undergone several significant developments and, by now, incorporates a complementary theory of 'Kolyvagin systems' of Mazur and Rubin [33]. There is also a theory of 'higher rank' Euler and Kolyvagin systems for representations $T$ endowed with the action of a commutative Gorenstein $\mathbb{Z}_p$-order $\mathcal{A}$ that was recently developed, following ideas of Rubin [45], Perrin-Riou [43] and Mazur and Rubin [34], by Sakamoto and the present authors in [10].

All of these theories are, however, intrinsically 'commutative' in nature (as systems comprise families of elements in exterior powers of cohomology modules over abelian extensions of $K$) and are therefore not well-suited to the finer study of leading term conjectures relevant to non-abelian Galois extensions. We remind the reader that such conjectures include the 'non-commutative Tamagawa number conjecture' of Fukaya and Kato [25] and the 'main conjecture of non-commutative Iwasawa theory' for elliptic curves without complex multiplication formulated by Coates et al [17] as well as, in a more classical setting, Chinburg's '$\Omega(3)$-Conjecture' in Galois module theory [16] and the 'non-abelian Brumer-Stark Conjecture' formulated, independently, by Nickel [41] and the first author [5].

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With this in mind, our aim is to take the first steps in the development of a usable theory of Euler systems (of arbitrary rank) in the setting of arbitrary Galois extensions of $K$. For this, we fix an algebraic closure $\mathbb{Q}^c$ of $\mathbb{Q}$, a Galois extension $\mathcal{K}$ of $K$ and for each $\mathbb{Q}^c$-valued character $\chi$ of $\mathrm{Gal}(\mathcal{K}/K)$ with open kernel, a representation $\mathrm{Gal}(\mathcal{K}/K) \to \mathrm{GL}_{\chi(1)}(\mathbb{Q}^c)$. To each such collection of representations, the methods of [13] associate a functorial family of 'reduced exterior powers' and 'reduced Rubin lattices' relative to the group rings over $\mathcal{A}$ of finite Galois extensions of $K$ in $\mathcal{K}$. We use these constructions to define, for each non-negative integer $r$, a natural notion of non-commutative Euler system (or 'nc-Euler system' for short) of rank $r$ relative to the $\mathcal{A}$-module $T$ and extension $\mathcal{K}/K$.

There are then at least two natural questions: do (non-zero) nc-Euler systems exist in any general setting and, if they do, what arithmetic properties do they have? These issues are the principle interest of the present article and we now discuss them in reverse order.

To give a brief statement of our main result concerning the properties of nc-Euler systems we use the phrase 'standard hypotheses' to refer to the basic conditions required for the commutative theory of Euler and Kolyvagin systems, as developed for systems of core rank one by Mazur and Rubin in [33] and for systems of arbitrary core rank by Sakamoto and the present authors in [11]. We use Selmer groups relative to the 'canonical' Selmer structure of Mazur and Rubin (cf. Remark 2.18). Then a precise version of the following result is stated as Theorem 2.15 and Corollary 2.16 and its proof relies on the main result of [11].

**Theorem A.** *If $p$ is odd and $T$ and $\mathcal{K}$ satisfy standard hypotheses, then the values of non-commutative Euler systems annihilate, as Galois modules, the Selmer and Tate-Shafarevich groups of $T$ over finite Galois extensions of $K$ in $\mathcal{K}$.*

This result confirms that nc-Euler systems encode the same arithmetic information over finite, possibly non-abelian, Galois extensions as do classical Euler systems over the base field. Given the key role Euler systems have played in the study of leading term conjectures, this offers hope that nc-Euler systems (when they exist) can contribute towards the resolution of leading term conjectures relevant to non-abelian Galois extensions.

However, just as in the classical commutative case, in any given setting it is a priori far from clear that there should exist *any* non-trivial nc-Euler systems. As a first general result in this direction, we shall therefore prove that, under relatively mild hypotheses on $T$ and $\mathcal{K}/K$, there exist (non-trivial) nc-Euler systems of rank depending explicitly on $T$ that determine higher non-commutative Fitting invariants associated to the cohomology groups of $T$ over Galois extension of $K$ in $\mathcal{K}$. The relevant results are proved as Theorems 2.19, 2.22 and 4.7 and Proposition 4.6 and the construction that underlies them is motivated, broadly speaking, by an approach of Kato to the formulation of generalized main conjectures in (commutative) Iwasawa theory that is described in [30].

Despite any intrinsic interest that this construction has, it still of course leaves open the key issue of whether, in the case that $T$ is a full lattice in the $p$-adic realisation of a motive, there should exist systems explicitly linked to the leading terms of Artin-twists of the corresponding $L$-series? This question is considered in detail in [14], but, as a straightforward consequence of results that are derived in §5.2 from Theorem 2.19 in the case that $T = \mathbb{Z}_p(1)$ and $\mathcal{A} = \mathbb{Z}_p$, we can at least give an affirmative, and unconditional, answer to it in an important special case.

To state this result, we assume to be given for each natural number $n$ a choice of primitive $n$-th root of unity $\zeta_n$ in $\mathbb{Q}^c$ such that $\zeta_m = (\zeta_n)^{n/m}$ for all divisors $m$ of $n$. We then recall that for any finite abelian totally real extension $F$ of $\mathbb{Q}$ of conductor $f$ (so that $F \subset \mathbb{Q}(\zeta_f)$) the classical cyclotomic element of $F$ is the element $\epsilon_F := \mathrm{Norm}_{\mathbb{Q}(\zeta_f)/F}(1-\zeta_f)$ of $F^\times$. In the following result we use for each finite Galois extension $F$ of $\mathbb{Q}$, with $\mathcal{G}_F := \mathrm{Gal}(F/\mathbb{Q})$, both the ideal $\delta(\mathbb{Z}[\mathcal{G}_F])$ of the centre of $\mathbb{Z}[\mathcal{G}_F]$ and reduced exterior powers ${\bigwedge}^1_{\mathbb{Q}[\mathcal{G}_F]}(-)$ that are defined in [13]. For each finite set $\Sigma$ of places of $\mathbb{Q}$ that contains $\infty$ we write $\mathcal{O}_{F,\Sigma}$ for the subring of $F$ comprising elements integral at all places of $F$ that do not restrict to give a place in $\Sigma$. For an integer $a$ and complex character $\chi$ of $\mathcal{G}_F$ we write $L^a_\Sigma(\chi,0)$ for the coefficient of $z^a$ in the Laurent expansion at $z=0$ of the $\Sigma$-truncated Artin $L$-series $L_\Sigma(\chi,z)$. We fix an isomorphism of fields $\mathbb{C}\cong\mathbb{C}_p$ (that we do not explicitly indicate) and use it to regard each $\chi$ as taking values in $\mathbb{C}_p$. We also fix (compatible) embeddings of $\mathbb{Q}^c$ into $\mathbb{C}$ and $\mathbb{Q}^c_p$ and use the restriction of these embeddings to define an archimedean place $w_{\infty,F}$ and a $p$-adic place $w_{p,F}$ of each field $F$ as above. Finally, for each such field we write $S(F)$ for the set of places of $\mathbb{Q}$ comprising $\infty, p$ and the primes that ramify in $F$.

**Theorem B.** *If $p$ is odd, then there exists a rank one nc-Euler system $\varepsilon^{\mathrm{cyc}} = (\varepsilon^{\mathrm{cyc}}_F)_F$ for $\mathbb{Z}_p(1)$ over the maximal totally real extension of $\mathbb{Q}$ in $\mathbb{Q}^c$ that has the following properties. In these claims, we write $F$ for an arbitrary finite totally real Galois extension of $\mathbb{Q}$ in $\mathbb{Q}^c$.*

(i) *If $\mathcal{G}_F$ is abelian, then $\varepsilon^{\mathrm{cyc}}_F$ is equal to $\epsilon_F$ if $p$ ramifies in $F$, respectively to $(\epsilon_F)^{1-\sigma_{p,F}}$ if $p$ is unramified in $F$ and $\sigma_{p,F}$ is its inverse Frobenius automorphism in $\mathcal{G}_F$.*

(ii) *For all $\varphi \in \mathrm{Hom}_{\mathcal{G}_F}(\mathcal{O}^\times_{F,S(F)}, \mathbb{Z}[\mathcal{G}_F])$ and $x \in \delta(\mathbb{Z}[\mathcal{G}_F])$, the element $x\cdot\big({\bigwedge}^1_{\mathbb{Q}[\mathcal{G}_F]}\varphi\big)(\varepsilon^{\mathrm{cyc}}_F)$ belongs to $\mathbb{Z}_p[\mathcal{G}_F]$ and annihilates $\mathbb{Z}_p\otimes_{\mathbb{Z}}\mathrm{Cl}(\mathcal{O}_F[1/\ell])$ for every prime $\ell$ in $S(F)$.*

(iii) *For every irreducible complex character $\chi$ of $\mathcal{G}_F$ one has*

$$({\bigwedge}^{\chi(1)}_{\mathbb{C}_p}\mathrm{Reg}^\chi_F)(e_\chi(\varepsilon^{\mathrm{cyc}}_F)) = u_{F,\chi}\cdot L^{\chi(1)}_{S(F)}(\check{\chi},0)\cdot e_\chi({\bigwedge}^1_{\mathbb{C}_p[\mathcal{G}_F]}(w_{\infty,F}-w_{p,F})).$$

*Here $\mathrm{Reg}^\chi_F$ is a canonical isomorphism of $\mathbb{C}_p$-modules induced by the Dirichlet regulator map (see §5.2.2), $e_\chi$ is the idempotent $\chi(1)|\mathcal{G}_F|^{-1}\cdot\sum_{g\in\mathcal{G}_F}\chi(g)g^{-1}$ of $\mathbb{C}[\mathcal{G}_F]$ and $\check{\chi}$ is the contragredient of $\chi$. In addition, one has $u_{F,\chi}=2$ if $\chi(1)=1$ and, if $\chi(1)>1$, then $u_{F,\chi}$ is a uniquely specified element of $\mathbb{C}^\times_p$ such that*

$$\prod\nolimits_{\omega\in\mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})} u_{F,\chi^\omega}\in\mathbb{Z}^\times_p,$$

*where $\mathbb{Q}(\chi)$ is the subfield of $\mathbb{C}$ generated by $\{\chi(g) : g\in\mathcal{G}_F\}$.*

Since each map $\mathrm{Reg}^\chi_F$ is injective, claim (iii) of this result implies, for all $F$ and $\chi$, that

$$e_\chi(\varepsilon^{\mathrm{cyc}}_F)\neq 0 \Longleftrightarrow L^{\chi(1)}_{S(F)}(\check{\chi},0)\neq 0.$$

In addition, if $\mathbb{Q}(\chi)=\mathbb{Q}$, as is automatically the case if $\mathcal{G}_F/\ker(\chi)$ is isomorphic to a quotient of a symmetric group, then the displayed containment in claim (iii) implies $u_{F,\chi}$ belongs to $\mathbb{Z}^\times_p$ (and for other refinements see Remark 5.10).

The existence of an nc-Euler system interpolating the leading terms at 0 of Artin $L$-series over a number field $k$ (as constructed above for $k=\mathbb{Q}$) can be combined with the techniques developed for proving Theorem A to give a method for studying the equivariant Tamagawa number conjecture for $\mathbb{G}_m$ over finite Galois extensions of $k$. This aspect is discussed in detail in the companion article [14]. However, the construction of such systems is also potentially of much broader interest in relation to the strategies described by Huber and Kings in [27, §3.3] and by Fukaya and Kato in [25, §2.3.5] to study the general case of the equivariant Tamagawa number conjecture. To explain this point, we let $E$ be an elliptic curve with complex multiplication by an imaginary quadratic field $k$ in $\mathbb{Q}^c$. Then the most effective method for investigating the Birch and Swinnerton-Dyer Conjecture for $E$ over abelian extensions of $k$ is to use Euler systems obtained by 'twisting' the elliptic unit Euler system that interpolates the values at 0 of first derivatives of Dirichlet $L$-series over $k$ by the $p$-adic Tate module $T_p(E)$ of $E$ (see, for example, the recent results of [15]). This approach relies crucially on the fact that the action of $\mathrm{Gal}(\mathbb{Q}^c/k)$ on each $T_p(E)$ factors through an abelian extension of $k$. From this perspective, the existence of nc-Euler systems that interpolate leading terms at 0 of Artin $L$-series over $\mathbb{Q}$ now raises the prospect that an analogous (non-commutative) twisting by $p$-adic Tate modules could be used to construct Euler systems that are suitable for the study of rational elliptic curves without complex multiplication.

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## 2. The general theory

The results obtained here depend crucially on the algebraic theory developed in the companion article [13].

### 2.1. The definition of non-commutative Euler systems.

2.1.1. In this section we fix a number field $K$, with algebraic closure $K^c$, and set $G_K := \mathrm{Gal}(K^c/K)$.

For each non-archimedean place $v$ of $K$ we write $\sigma_v$ for the *inverse* of a fixed choice of Frobenius automorphism of $v$ in the Galois group of the maximal extension of $K$ in $K^c$ that is unramified at $v$.

We also write $\mathrm{Ir}_p(K)$ for the set of distinct irreducible $\mathbb{Q}_p^c$-valued characters of $G_K$ that have open kernel.

For a Galois extension $\mathcal{K}$ of $K$ in $K^c$ we write $\Omega(\mathcal{K}/K)$ for the set of finite Galois extensions of $K$ in $\mathcal{K}$.

For $F$ in $\Omega(K^c/K)$ we set $\mathcal{G}_F := \mathrm{Gal}(F/K)$ and write $\mathrm{Ir}_p(\mathcal{G}_F)$ for the subset of $\mathrm{Ir}_p(K)$ comprising characters that factor through the restriction map $G_K \to \mathcal{G}_F$.

For $\chi$ in $\mathrm{Ir}_p(K)$ we write $K(\chi)$ for the subfield of $K^c$ that is fixed by $\ker(\chi)$ and $n_\chi$ for the exponent of $\mathcal{G}_{K(\chi)}$. We also write $E_\chi$ for the subfield of $\mathbb{Q}_p^c$ generated by a choice of primitive $n_\chi$-th root of unity and, following [4], we fix a representation

$$\rho_\chi : \mathcal{G}_{K(\chi)} \to \mathrm{GL}_{\chi(1)}(E_\chi)$$

of character $\chi$. For $F$ in $\Omega(\mathcal{K}/K)$ we write $E_F$ for the composite of the fields $E_\chi$ as $\chi$ runs over $\mathrm{Ir}_p(\mathcal{G}_F)$. Then, for any subfield $L$ of $\mathbb{Q}_p^c$, the discussion of [13, Rem. 4.9] shows that the fixed choice of representation $\rho_\chi$ for each $\chi$ in $\mathrm{Ir}_p(\mathcal{G}_F)$ induces an isomorphism of $LE_F$-algebras

$$(LE_F)[\mathcal{G}_F] \cong \prod\nolimits_{\chi \in \mathrm{Ir}_p(\mathcal{G}_F)} \mathrm{M}_{\chi(1)}(LE_F),$$

and hence determines a natural choice of the auxiliary data needed to define reduced exterior powers, reduced Rubin lattices and reduced determinant functors over the semisimple algebra $L[\mathcal{G}_F]$ (for details see [13, §4.2, §4.4 and §5]). We assume throughout the sequel, and without further explicit comment, that for any finite set of extensions $\{L_i\}_{i\in I}$ of $K$ in $K^c$ and any $F$ in $\Omega(K^c/K)$ the constructions of [13] over the semisimple algebra $\prod_{i\in I} L_i[\mathcal{G}_F]$ are made relative to this choice of data.

2.1.2. We next assume to be given a finite extension $\mathcal{Q}$ of $\mathbb{Q}_p$ in $\mathbb{Q}_p^c$, with valuation ring $\mathcal{O}$. For an $\mathcal{O}$-module $M$ we set $M^* := \mathrm{Hom}_{\mathcal{O}}(\mathcal{A}, \mathcal{O})$ and $M^\vee := \mathrm{Hom}_{\mathcal{O}}(\mathcal{A}, \mathcal{Q}/\mathcal{O})$.

We fix an $\mathcal{O}$-order $\mathcal{A}$ that spans a finite-dimensional semisimple commutative $\mathcal{Q}$-algebra $A$ and also satisfies the following condition.

**Hypothesis 2.1.** *$\mathcal{A}^*$ is a free module of rank one with respect to the natural action of $\mathcal{A}$.*

**Remark 2.2.** If Hypothesis 2.1 is satisfied, then for each field $F$ in $\Omega(K^c/K)$ the $\mathcal{O}$-order $\mathcal{A}[\mathcal{G}_F]$ is a one-dimensional Gorenstein ring. In particular, in each such case, the group $\mathrm{Ext}^1_{\mathcal{A}[\mathcal{G}_F]}(M, \mathcal{A}[\mathcal{G}_F])$ vanishes for every finitely generated $\mathcal{A}[\mathcal{G}_F]$-module $M$ that is $\mathcal{O}$-torsion-free. (For more details see either [12, §A.3] or [18, §37].)

We also assume to be given a continuous $\mathcal{O}$-representation $T$ of $G_K$ that satisfies the following condition.

**Hypothesis 2.3.** *$T$ is endowed with a (commuting) action of $\mathcal{A}$ with respect to which it is projective.*

We write $S_\infty(K)$ and $S_p(K)$ for the sets of archimedean and $p$-adic places of $K$ and $S_{\mathrm{bad}}(T)$ for the (finite) set of places of $K$ at which $T$ has bad reduction. For each field $F$ in $\Omega(K^c/K)$ we write $S_{\mathrm{ram}}(F/K)$ for the set of places of $K$ that ramify in $F$ and then consider the finite set of places of $K$ given by

$$S(F) = S(T, F) := S_\infty(K) \cup S_p(K) \cup S_{\mathrm{bad}}(T) \cup S_{\mathrm{ram}}(F/K).$$

Now for each $F$ in $\Omega(K^c/K)$ the induced representation

$$T_F := \mathrm{Ind}_{G_K}^{G_F}(T)$$

is naturally a module over $\mathcal{A}[\mathcal{G}_F]$, and identifies with the tensor product $\mathcal{A}[\mathcal{G}_F] \otimes_{\mathcal{A}} T$ upon which $\mathcal{A}[\mathcal{G}_F]$ acts via left multiplication on the first factor whilst $G_K$ acts by

$$\sigma \cdot (a \otimes t) := a\overline{\sigma}^{-1} \otimes \sigma t \ (\sigma \in G_K,\ a \in \mathcal{A}[\mathcal{G}_F],\ \text{and } t \in T),$$

where $\overline{\sigma} \in \mathcal{G}_F$ is the image of $\sigma$.

In particular, for each place $v$ of $K$ outside $S(F)$ there is a natural action of the automorphism $\sigma_v$ on $T_F$ (that commutes with the action of $\mathcal{A}[\mathcal{G}_F]$).

Hypothesis 2.3 implies that for each integer $a$ the representation $T(a) := T \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(a)$ is a projective $\mathcal{A}$-module and hence that the induced representation

$$T(a)_F \cong (T_F)(a) \cong \mathcal{A}[\mathcal{G}_F] \otimes_{\mathcal{A}} T(a)$$

is a projective $\mathcal{A}[\mathcal{G}_F]$-module.

Hypotheses 2.1 and 2.3 combine to imply that the Kummer dual representations $T^*(1)$ and $T^*(1)_F$ are respectively projective modules over $\mathcal{A}$ and $\mathcal{A}[\mathcal{G}_F]$.

We shall also use the $A$-linear representations $V := \mathcal{Q} \otimes_{\mathcal{O}} T$ and $V^*(1) := \mathcal{Q} \otimes_{\mathcal{O}} T^*(1)$ and for each $F$ in $\Omega(\mathcal{K}/K)$ also $V_F := \mathcal{Q} \otimes_{\mathcal{O}} T_F$ and $V^*(1)_F := \mathcal{Q} \otimes_{\mathcal{O}} T^*(1)_F$.

2.1.3. We fix a Wedderburn decomposition $A = \prod_{i \in I} L_i$ where each $L_i$ is a finite extension of $\mathbb{Q}_p$ in $\mathbb{Q}_p^c$ that contains $\mathcal{Q}$. Then for each place $v$ of $K$ outside $S(F)$ the reduced norm

$$\mathrm{Nrd}_{A[\mathcal{G}_F]}(1 - \sigma_v \mid V^*(1)_F)$$

over the semisimple algebra $A[\mathcal{G}_F] = \prod_{i \in I} L_i[\mathcal{G}_F]$ can be computed as the reduced norm of an endomorphism of the projective $\mathcal{A}[\mathcal{G}_F]$-module $T^*(1)_F$ and so belongs to $\xi(\mathcal{A}[\mathcal{G}_F])$.

We also note that for any pair of fields $F$ and $F'$ in $\Omega(\mathcal{K}/K)$ with $F \subseteq F'$, the observation of [13, Rem. 4.5] applies to each simple component of $A[\mathcal{G}_F]$ to imply that the natural corestriction map

$$\mathrm{Cor}_{F'/F} : H^1(\mathcal{O}_{F',S(F')}, T) \to H^1(\mathcal{O}_{F,S(F')}, T)$$

induces a homomorphism of $\zeta(A)$-modules

$$\mathrm{Cor}^a_{F'/F} : {\bigwedge}^a_{A[\mathcal{G}_{F'}]} H^1(\mathcal{O}_{F',S(F')}, V) \to {\bigwedge}^a_{A[\mathcal{G}_F]} H^1(\mathcal{O}_{F,S(F')}, V).$$

In the sequel we write $x \mapsto x^\#$ for the involution of $\zeta(A[\mathcal{G}_F])$ that is induced by restricting the $A$-linear anti-involution of $A[\mathcal{G}_F]$ that inverts elements of $\mathcal{G}_F$.

We can now introduce the definition of non-commutative ($p$-adic) Euler systems.

**Definition 2.4.** Let $a$ be a non-negative integer. Then a *non-commutative Euler system* (or *nc-Euler system* for short) *of rank* $a$ for the pair $(T, \mathcal{K})$ is a family of elements

$$c = (c_F)_F \in \prod\nolimits_{F \in \Omega(\mathcal{K}/K)} {\bigcap}^a_{\mathcal{A}[\mathcal{G}_F]} H^1(\mathcal{O}_{F,S(F)}, T)$$

with the property that for every $F$ and $F'$ in $\Omega(\mathcal{K}/K)$ with $F \subset F'$ one has

$$\mathrm{Cor}^a_{F'/F}(c_{F'}) = \left( \prod\nolimits_{v \in S(F') \setminus S(F)} \mathrm{Nrd}_{A[\mathcal{G}_F]}(1 - \sigma_v \mid V^*(1)_F)^\# \right)(c_F) \tag{2.1.1}$$

in ${\bigwedge}^a_{A[\mathcal{G}_F]} H^1(\mathcal{O}_{F,S(F')}, V)$. We write $\mathrm{ES}_a(T, \mathcal{K})$ for the set of nc-Euler systems of rank $a$ for the pair $(T, \mathcal{K})$.

**Remark 2.5.** If $\mathcal{K}/K$ is abelian, then [13, Rem. 4.18] implies that the above definition of an nc-Euler system of rank $a$ for $(T, \mathcal{K})$ agrees with that given in [12, Def. 2.3] (with $T$ replaced by $T^*(1)$). In particular, if $\mathcal{K}/K$ is abelian and $a = 1$, then the above definition recovers the classical definition of Euler systems for $p$-adic representations that is given by Rubin in [46, Def. 2.1.1].

**Remark 2.6.** In all cases, it is clear that the set $\mathrm{ES}_a(T,\mathcal{K})$ is an abelian group with respect to the addition of systems that is defined by $c_1 + c_2 := (c_{1,F} + c_{2,F})_{F\in\Omega(\mathcal{K}/K)}$. This group is also endowed with a natural action of the inverse limit algebra

$$\xi(\mathcal{A}[[\mathrm{Gal}(\mathcal{K}/K)]]) := \varprojlim_{L\in\Omega(\mathcal{K}/K)}\xi(\mathcal{A}[\mathcal{G}_L])$$

where the (surjective) transition morphisms are induced by the natural projection maps $\mathcal{A}[\mathcal{G}_{L'}] \to \mathcal{A}[\mathcal{G}_L]$ for $L \subseteq L'$ (and [13, Lem. 3.2(v)]). More precisely, for $c \in \mathrm{ES}_a(T,\mathcal{K})$ and $\lambda = (\lambda_L)_L \in \xi(\mathcal{A}[[\mathrm{Gal}(\mathcal{K}/K)]])$ one obtains a well-defined element of $\mathrm{ES}_a(T,\mathcal{K})$ by setting

$$\lambda(c) := (\lambda_F(c_F))_{F\in\Omega(\mathcal{K}/K)},$$

and, via the assignment $(\lambda, c) \mapsto \lambda(c)$, the group $\mathrm{ES}_a(T,\mathcal{K})$ is a $\xi(\mathcal{A}[[\mathrm{Gal}(\mathcal{K}/K)]])$-module.

**Remark 2.7.** If $T = \mathbb{Z}_p(1)$, then Kummer theory identifies $H^1(\mathcal{O}_{F,S(F)}, T)$ with the pro-$p$ completion of the group of $S(F)$-units of $F$. In this classical setting, it is possible to develop a finer version of the theory that we describe below by considering compatible families of elements that are defined just as above but with $S(F)$ replaced by the subset comprising $S_\infty(K)$ and all places that ramify in $F$. Such 'non-commutative Euler systems for $\mathbb{G}_m$' are studied in the supplementary article [14].

2.1.4. In the sequel it is often convenient to assume $T$ and $\mathcal{K}$ satisfy the following hypothesis.

**Hypothesis 2.8.** For all fields $F$ in $\Omega(\mathcal{K}/K)$

(i) $H^0(\mathcal{O}_{F,S(F)}, T)$ vanishes, and
(ii) $H^1(\mathcal{O}_{F,S(F)}, T)$ is $\mathcal{O}$-torsion-free.

**Remark 2.9.** This hypotheses is automatically satisfied in many cases of interest. (For example, if $p$ is odd and $\mathcal{K}$ is the maximal totally real extension of $K = \mathbb{Q}$, then it is satisfied in the context of Remark 2.7). In general, if Hypothesis 2.8(ii) fails to be valid but there exists a finite non-empty set of places $\Sigma$ of $K$ that is unramified in $\mathcal{K}$, then the '$\Sigma$-modification' construction described in [12, §2.3] defines a canonical torsion-free submodule $H^1_\Sigma(\mathcal{O}_{F,S(F)}, T)$ of $H^1(\mathcal{O}_{F,S(F)}, T)$ for each $F$ in $\Omega(\mathcal{K}/K)$. By systematically replacing groups of the form $H^1(\mathcal{O}_{F,S(F)}, T)$ by $H^1_\Sigma(\mathcal{O}_{F,S(F)}, T)$ in what follows, one can establish an analogue of the theory below without assuming the validity of Hypothesis 2.8(ii). However, since this extended theory is obtained in just the same way, we prefer to avoid the extra technicalities and do not discuss it further.

**Lemma 2.10.** *Assume $T$ and $\mathcal{K}$ satisfy Hypothesis 2.8. Then for every pair of fields $F$ and $F'$ in $\Omega(\mathcal{K}/K)$ with $F \subset F'$ and every natural number $a$ the following claims are valid.*

(i) *The restriction map $H^1(\mathcal{O}_{F,S(F')}, T) \to H^1(\mathcal{O}_{F',S(F')}, T)$ identifies $H^1(\mathcal{O}_{F,S(F')}, T)$ with the submodule of $\mathrm{Gal}(F'/F)$-invariant elements of $H^1(\mathcal{O}_{F',S(F')}, T)$.*

(ii) *The corestriction map $\mathrm{Cor}^a_{F'/F}$ restricts to give a homomorphism of $\xi(\mathcal{A}[\mathcal{G}_F])$-modules*

$$\bigcap\nolimits^a_{\mathcal{A}[\mathcal{G}_{F'}]} H^1(\mathcal{O}_{F',S(F')}, T) \to \bigcap\nolimits^a_{\mathcal{A}[\mathcal{G}_F]} H^1(\mathcal{O}_{F,S(F')}, T).$$

(iii) *The inflation map $H^1(\mathcal{O}_{F,S(F)}, T) \to H^1(\mathcal{O}_{F,S(F')}, T)$ induces an identification*

$$\bigcap\nolimits^a_{\mathcal{A}[\mathcal{G}_F]} H^1(\mathcal{O}_{F,S(F)}, T) = \big(\bigcap\nolimits^a_{\mathcal{A}[\mathcal{G}_F]} H^1(\mathcal{O}_{F,S(F')}, T)\big) \cap \big(\bigwedge\nolimits^a_{A[\mathcal{G}_F]} H^1(\mathcal{O}_{F,S(F)}, V)\big).$$

*Proof.* We set $\Delta := \mathrm{Gal}(F'/F)$, $X_1 := H^1(\mathcal{O}_{F,S(F')}, T)$ and $X_2 := H^1(\mathcal{O}_{F',S(F')}, T)$.

Hypothesis 2.8(i) implies that the complex $\mathsf{R}\Gamma(\mathcal{O}_{F',S(F')}, T)$ is acyclic in degrees less than one. Given this fact, claim (i) follows from the fact that the fixed point functor $M \mapsto \mathrm{Hom}_{\mathcal{O}[\Delta]}(\mathcal{O}, M) = H^0(\Delta, M)$ is left exact and that there is a canonical isomorphism

$$\mathsf{R}\,\mathrm{Hom}_{\mathcal{O}[\Delta]}\big(\mathcal{O}, \mathsf{R}\Gamma(\mathcal{O}_{F',S(F')}, T)\big) \cong \mathsf{R}\Gamma(\mathcal{O}_{F,S(F')}, T)$$

in $\mathsf{D}(\mathcal{O}[\mathcal{G}_F])$.

We next recall a general fact: for any commutative ring $R$, finite group $G$ and left $R[G]$-module $M$, there is a natural isomorphism of $R[G]$-modules

$$\mathrm{Hom}_R(M, R) \xrightarrow{\sim} \mathrm{Hom}_{R[G]}(M, R[G]);\ f \mapsto \sum\nolimits_{\sigma \in G} f(\sigma(-))\sigma^{-1}, \tag{2.1.2}$$

where $G$ acts on the dual modules $\mathrm{Hom}_R(M, R)$ and $\mathrm{Hom}_{R[G]}(M, R[G])$ via the rules

$$(\sigma \cdot f)(m) := f(\sigma^{-1}m)\ (\sigma \in G,\ f \in \mathrm{Hom}_R(M, R),\ m \in M),$$

and

$$(\sigma \cdot \theta)(m) := \theta(m)\sigma^{-1}\ (\sigma \in G,\ \theta \in \mathrm{Hom}_{R[G]}(M, R[G]),\ m \in M).$$

Turning to claim (ii) we note first that claim (i) combines with Hypothesis 2.8(ii) to imply the cokernel of the restriction map $\varrho_{F'/F} : X_1 \to X_2$ is $\mathcal{O}$-torsion-free. (This is because if $x$ is any element of $X_2$ such that for some $n > 1$ one has $(\delta - 1)(n \cdot x) = 0$ for all $\delta \in \Delta$, then $n((\delta - 1)(x)) = 0$ and hence, since $X_2$ is torsion-free, also $(\delta - 1)(x) = 0$ for all $\delta \in \Delta$.)

The assumption that $\mathcal{A}$ is Gorenstein therefore implies that $\mathrm{Ext}^1_{\mathcal{A}}\big(\mathrm{cok}(\varrho_{F'/F}), \mathcal{A}\big)$ vanishes (cf. Remark 2.2) and hence that the composite homomorphism

$$\begin{aligned}\varrho^*_{F'/F} : \mathrm{Hom}_{\mathcal{A}[\mathcal{G}_{F'}]}(X_2, \mathcal{A}[\mathcal{G}_{F'}]) &\cong \mathrm{Hom}_{\mathcal{A}}(X_2, \mathcal{A}) \\ &\xrightarrow{\mathrm{Hom}_{\mathcal{A}}(\varrho_{F'/F}, \mathcal{A})} \mathrm{Hom}_{\mathcal{A}}(X_1, \mathcal{A}) \cong \mathrm{Hom}_{\mathcal{A}[\mathcal{G}_F]}(X_1, \mathcal{A}[\mathcal{G}_F])\end{aligned}$$

is surjective, where the two isomorphisms are as in (2.1.2).

One can also check that for each $\theta$ in $\mathrm{Hom}_{\mathcal{A}[\mathcal{G}_{F'}]}(X_2, \mathcal{A}[\mathcal{G}_{F'}])$ the diagram

$$\begin{array}{ccc} X_2 & \xrightarrow{\theta} & \mathcal{A}[\mathcal{G}_{F'}] \\ {\scriptstyle \mathrm{Cor}_{F'/F}}\big\downarrow & & \big\downarrow \\ X_1 & \xrightarrow{\varrho^*_{F'/F}(\theta)} & \mathcal{A}[\mathcal{G}_F] \end{array} \tag{2.1.3}$$

commutes, where the unlabelled arrow is the natural projection map. This diagram in turn implies that for every subset $\{\theta_i\}_{1 \le i \le a}$ of $\mathrm{Hom}_{\mathcal{A}[\mathcal{G}_{F'}]}(X_2, \mathcal{A}[\mathcal{G}_{F'}])$ and every element $x$ of $\bigwedge^a_{A[\mathcal{G}_{F'}]} H^1(\mathcal{O}_{F,S(F)}, V) = \bigwedge^a_{A[\mathcal{G}_{F'}]}(\mathcal{Q} \otimes_{\mathcal{O}} X_2)$ one has

$$\big(\wedge_{i=1}^{i=a} \varrho^*_{F'/F}(\theta_i)\big)\big(\mathrm{Cor}^a_{F'/F}(x)\big) = \pi_{F'/F}\big((\wedge_{i=1}^{i=a}\theta_i)(x)\big) \tag{2.1.4}$$

where $\pi_{F'/F}$ is the natural projection map $\zeta(A[\mathcal{G}_{F'}]) \to \zeta(A[\mathcal{G}_F])$.

In particular, this equality combines with the surjectivity of $\varrho^*_{F'/F}$ to imply if $x$ belongs to $\bigcap^a_{\mathcal{A}[\mathcal{G}_{F'}]} X_2$, then $\mathrm{Cor}^a_{F'/F}(x)$ belongs to $\bigcap^a_{\mathcal{A}[\mathcal{G}_F]} X_1$, as required to prove claim (ii).

To prove claim (iii) we use the canonical exact sequence

$$0 \to H^1(\mathcal{O}_{F,S(F)}, T) \xrightarrow{\iota_{F,S,S'}} H^1(\mathcal{O}_{F,S(F')}, T) \to \bigoplus\nolimits_{w \in (S(F')\setminus S(F))_F} H^1_{/f}(F_w, T),$$

in which $H^1_{/f}(F_w, T)$ denotes the cokernel of the inflation map $H^1(F_w^{\rm ur}/F_w, T) \to H^1(F_w, T)$, where $F_w^{\rm ur}$ is the maximal unramified extension of $F_w$ in $F_w^c$.

In addition, since each $\mathcal{O}$-module $H^1_{/f}(F_w, T)$ is free (by [46, Lem. 1.3.5(ii)]) the $\mathcal{O}$-module $\mathrm{cok}(\iota_{F,S,S'})$ is also free and so the group $\mathrm{Ext}^1_{\mathcal{A}[\mathcal{G}_F]}\big(\mathrm{cok}(\iota_{F,S,S'}), \mathcal{A}[\mathcal{G}_F]\big)$ vanishes (by Remark 2.2). Given this last fact, the identification in claim (iii) follows immediately from the final assertion of [13, Th. 4.19(iv)] with $\iota = \iota_{F,S,S'}$. □

2.2. **Statement of the main results.** In this section we present our main results concerning the general theory of non-commutative Euler systems (and, in particular, state a precise version of Theorem A).

2.2.1. We first consider $p$-adic representations satisfying conditions that are standard in the theory of higher rank Euler and Kolyvagin systems, as developed by Sakamoto and the present authors in [11] following the approach initiated by Mazur and Rubin in [33] and [34]. In particular, by combining the main result of [11] with Morita-equivalence type arguments, we show that nc-Euler systems for this class of representations provide annihilators for the Galois modules that arise from Selmer groups over non-abelian Galois extensions. This result constitutes a precise version of Theorem A in the Introduction.

To explain its setting, we assume to be given a natural number $n$, a commutative Gorenstein $\mathcal{O}$-order $\mathcal{A}$ in a $\mathcal{Q}$-algebra $A$ and a homomorphism of $\mathcal{O}$-algebras

$$\varrho : \mathcal{O}[[G_K]] \to \mathrm{M}_n(\mathcal{A}) \tag{2.2.1}$$

that has all of the following properties:

- ($\mathrm{R}_1$) the cokernel of $\varrho$ is finite;
- ($\mathrm{R}_2$) the kernel of the restriction of $\varrho$ to $G_K$ is an open normal subgroup $H(\varrho)$;
- ($\mathrm{R}_3$) $\mathrm{M}_n(A)$ is isomorphic to a subalgebra of $\mathcal{Q}[G_K/H(\varrho)]$.

We set $G(\varrho) := G_K/H(\varrho)$ and define the 'kernel field' $K(\varrho)$ of $\varrho$ to be the fixed field of $H(\varrho)$ in $K^c$ (so that $G(\varrho)$ identifies with $\mathrm{Gal}(K(\varrho)/K)$). Then, since $\mathcal{Q}[G(\varrho)]$ is semisimple, the condition ($\mathrm{R}_3$) implies $\mathrm{M}_n(A)$ is a direct factor of $\mathcal{Q}[G(\varrho)]$ and hence that there exists an idempotent $e(\varrho)$ of $\zeta(\mathcal{Q}[G(\varrho)])$ such that

$$\mathcal{Q} \cdot \mathrm{im}(\varrho) = \mathrm{M}_n(A) = e(\varrho) \cdot \mathcal{Q}[G(\varrho)], \tag{2.2.2}$$

where the first equality follows from ($\mathrm{R}_1$).

We then define an $\mathcal{O}$-order in $\mathcal{Q}[G(\varrho)]$ by setting

$$\mathcal{A}(\varrho) := \mathrm{M}_n(\mathcal{A}) \oplus \mathcal{O}[G(\varrho)](1 - e(\varrho))$$

and, noting that $\mathcal{O}[G(\varrho)]$ is a finite index submodule of $\mathcal{A}(\varrho)$, we write

$$d(\varrho) := \mathrm{exponent}\big(\mathcal{A}(\varrho)/\mathcal{O}[G(\varrho)]\big)$$

for the exponent of the (finite) quotient of $\mathcal{A}(\varrho)$ by $\mathcal{O}[G(\varrho)]$.

**Examples 2.11.** There are several ways in which a finite Galois extension $L$ of $K$, with $G = \mathcal{G}_L$, can give rise to representation of the above form.
(i) If $\mathcal{Q}[G]$ is split, then for each character $\psi$ in $\mathrm{Ir}_p(G)$ there is an associated representation $\varrho_\psi$ of the form (2.2.1) in which $n = \psi(1)$, $\mathcal{A} = \mathcal{O}$, $G(\varrho_\psi)$ is the quotient of $G$ by $\ker(\psi)$, $K(\varrho_\psi)$ is the fixed field of $\ker(\psi)$ in $L$, $e(\varrho_\psi) = e_\psi$ and $d(\varrho_\psi)$ is a divisor of $|G|/\psi(1)$.
(ii) If $G$ has an abelian Sylow $p$-subgroup and a normal $p$-complement (or, equivalently, $p$ does not divide the order of the commutator subgroup of $G$), then $\mathcal{A}$ is a direct product of matrix rings over commutative $R$-algebras (cf. Demeyer and Janusz [22, p. 390, Cor]). Hence, in this case, for some finite index set $I$ there is a direct product decomposition

$$\mathcal{O}[G] = \prod\nolimits_{i \in I} \mathrm{M}_{n_i}(\mathcal{S}_i) \tag{2.2.3}$$

in which each $\mathcal{S}_i$ is a commutative $\mathcal{O}$-order. In particular, for each fixed index $i$, there exists a representation $\varrho_i$ of the form (2.2.1) in which $n = n_i$, $\mathcal{A}$ is any choice of Gorenstein $\mathcal{O}$-order in $\mathcal{Q} \cdot \mathcal{S}_i$ that contains $\mathcal{S}_i$ (such as, for example, the integral closure of $\mathcal{S}_i$ in $\mathcal{Q} \cdot \mathcal{S}_i$), $K(\varrho)$ is a subfield of $L$ and $d(\varrho)$ is the exponent of the finite group $\mathcal{A}/\mathcal{S}_i$.
(iii) The group $G$ is said to be a 'Frobenius group' if it has a proper non-trivial subgroup $H$ such that $H \cap gHg^{-1} = \{1\}$ for all $g \in G \setminus H$, in which case $G$ has a unique normal subgroup $N$ and $G$ is a semidirect product $N \rtimes H$. In addition, if the order of $N$ is prime to $p$ and $H$ is abelian, then $G$ is of the form discussed in (ii) and the result [29, Prop. 2.13] of Johnston and Nickel implies that every order $\mathcal{S}_i$ in the decomposition (2.2.3) is Gorenstein (more precisely, there exists an index $i_0 \in I$ for which $n_{i_0} = 1$ and $\mathcal{S}_{i_0}$ is isomorphic to $\mathbb{Z}_p[H]$ and then for all $i \in I \setminus \{i_0\}$ the ring $\mathcal{S}_i$ is a discrete valuation ring). Hence, in any such case, every index $i$ in $I$ gives rise to a representation $\varrho_i$ of the form (2.2.1) in which $n = n_i$, $\mathcal{A} = \mathcal{S}_i$, $e(\varrho) \in \mathcal{O}[G]$ and $d(\varrho) = 1$.
(iv) If $G$ is abelian, then (as a special case of (iii)) the multiplication action of $G$ on $\mathcal{O}[G]$ gives a representation $\varrho_G$ of the form (2.2.1) in which $n = 1$, $\mathcal{A} = \mathcal{O}[G]$, $K(\varrho_G) = L$, $e(\varrho_G) = 1$ and $d(\varrho_G) = 1$.

We now assume to be given a finitely generated free $\mathcal{O}$-module $T$ that is endowed with a continuous action of $G_K$. Writing $\mathcal{A}^n$ for the right $\mathrm{M}_n(\mathcal{A})$-module comprising row vectors over $\mathcal{A}$ of length $n$, we consider the associated left $\mathcal{A}[[G_K]]$-module

$$T\langle\varrho\rangle := \mathcal{A}^n \otimes_{\mathcal{O}} T$$

upon which the action is specified as follows: for $a$ in $\mathcal{A}$, $g$ in $G_K$, $v$ in $\mathcal{A}^n$ and $t$ in $T$ one has $(ag)(v \otimes t) = (a \cdot v \cdot \varrho(g^{-1})) \otimes g(t)$. We then consider the associated space

$$V\langle\varrho\rangle := A^n \otimes_{\mathcal{O}} T$$

with respect to the induced (left) action of $A[[G_K]]$.

We next fix a pro-$p$ abelian extension $\mathcal{L}$ of $K$ that satisfies the following hypothesis.

**Hypothesis 2.12.**

($\mathrm{E}_1$) *$\mathcal{L}$ is contained in $\mathcal{K}$.*

($\mathrm{E}_2$) *$\mathcal{L}$ contains the maximal $p$-extension of $K$ in the ray class fields modulo almost all prime ideals and, in addition, a $\mathbb{Z}_p$-extension of $K$ in which no finite place splits completely.*

($\mathrm{E}_3$) *the representation $T\langle\varrho\rangle$ satisfies Hypothesis 2.8 for every $F$ in $\Omega(\mathcal{L}/K)$.*

(E$_4$) *$\mathcal{L}$ is disjoint from the kernel field $K(\varrho)$ of $\varrho$.*

We fix a finite extension $F$ of $K$ in $\mathcal{L}$ and write $F(T,\varrho)$ for the minimal Galois extension of $K$ such that $G_{F(T,\varrho)}$ acts trivially on $T\langle\varrho\rangle$. With $K(1)$ denoting the Hilbert $p$-classfield of $K$, for each natural number $m$ we also set

$$F_{p^m} := K(\mu_{p^m}, (\mathcal{O}_K^\times)^{1/p^m})K(1) \quad \text{and} \quad F(T,\varrho)_m := F(T,\varrho)F_{p^m},$$

where $\mu_{p^m}$ denotes the group of $p^m$-th roots of unity in $K^c$ and $(\mathcal{O}_K^\times)^{1/p^m}$ the set of elements $u$ in $K^c$ for which $u^{p^m}$ belongs to $\mathcal{O}_K^\times$. We then obtain infinite Galois extensions of $K$ by setting

$$F_{p^\infty} := \bigcup_{m>0} F_{p^m} \text{ and } F(T,\varrho)_\infty := \bigcup_{m>0} F(T,\varrho)_m,$$

We write $\Bbbk$ for the residue field of $\mathcal{O}$ and consider the $\Bbbk[[G_k]]$-module

$$\overline{T\langle\varrho\rangle} := \Bbbk \otimes_{\mathcal{O}} T\langle\varrho\rangle.$$

We can now recall the hypotheses on $T$ and $F$ that are standard in the theory of higher rank Euler and Kolyvagin systems, as developed in [11] following earlier work of Mazur and Rubin in [33] and [34].

**Hypothesis 2.13.**

(H$_0$) *for almost all primes $\mathfrak{q}$ of $K$, the map $\mathrm{Fr}_{\mathfrak{q}}^{p^k} - 1$ is injective on $T\langle\varrho\rangle$ for every $k \geq 0$.*

(H$_1$) *the $\Bbbk[[G_K]]$-module $\overline{T\langle\varrho\rangle}$ is irreducible.*

(H$_2$) *there exists an element $\tau$ of $G_{F_{p^\infty}}$ for which the $\mathcal{O}$-module $T\langle\varrho\rangle/(\tau-1)T\langle\varrho\rangle$ is free of rank one.*

(H$_3$) *the groups $H^1(F(T,\varrho)_\infty/K, \overline{T\langle\varrho\rangle})$ and $H^1(F(T,\varrho)_\infty/K, \overline{T\langle\varrho\rangle}^\vee(1))$ both vanish.*

(H$_4$) *if $p=3$, then $\mathrm{Hom}_{\Bbbk[[G_K]]}(\overline{T\langle\varrho\rangle}, \overline{T\langle\varrho\rangle}^\vee(1))$ vanishes.*

**Remark 2.14.** These individual hypotheses are discussed in detail in [11, §3]. In particular, the hypotheses (H$_1$), (H$_2$), (H$_3$) and (H$_4$) respectively correspond to the hypotheses (H.1), (H.2), (H.3) and (H.4) that are used by Mazur and Rubin in [33] and hypothesis (H$_0$) corresponds to the assumption (b) in [33, Th. 3.2.4]. Further, in concrete cases, it is possible to analyse these hypotheses explicitly. To give a relatively straightforward example, we assume that $\mathcal{O} = \mathbb{Z}_p$ and $T = \mathbb{Z}_p(1)$ and that $\varrho = \varrho_\psi$ corresponds to an irreducible $\mathbb{C}_p$-representation $\psi$ of $G_K$ as in Example 2.11(i). We also set $L = K(\varrho)$ and $G = \mathcal{G}_L$ and assume that $K$ contains $\mu_p$. Then, in this case, (H$_0$) is automatically satisfied since no eigenvalue of $\mathrm{Fr}_{\mathfrak{q}}$ on $T\langle\varrho\rangle$ is a root of unity. In addition, since $\mu_p$ is contained in $K$, the $\Bbbk[[G_K]]$-module $\overline{T\langle\varrho\rangle}$ identifies with $\Bbbk^n$ upon which $G_K$ acts via $\psi$, and so (H$_1$) is satisfied whenever the modular reduction of $\psi$ remains irreducible (and for a discussion of representation-theoretic results in this direction see, for example, Fayers [23]). In a similar way, since $G_{F_{p^\infty}}$ acts trivially on $\mathbb{Z}_p(1)$, the condition (H$_2$) can be seen to be satisfied if there exists an element $g$ of $G$ that has order prime to $p$ and is such that 1 occurs as an eigenvalue of the matrix $\varrho(g)$ with multiplicity one. Next, to check (H$_3$), we assume that $p$ does not divide $|G|$, that $G = G_1 \times G_2$ with $G_2$ abelian and that $\psi = \psi_1 \times \psi_2$ with $\psi_1$ in $\mathrm{Ir}_p(G_1)$ and $\psi_2$ a non-trivial homomorphism $G_2 \to \mathbb{Q}_p^{c\times}$. Then, if $N$ denotes either $\overline{T\langle\varrho\rangle}$ or $\overline{T\langle\varrho\rangle}^\vee(1)$, the restriction map $H^1(F(T,\varrho)_\infty/K, N) \to H^1(F(T,\varrho)_\infty/L, N)$ is injective and, since the $\Bbbk[[G_L]]$-module

$N$ is isomorphic to the direct sum of $\psi(1)$ copies of $\mathbb{Z}/(p)$, the group $H^1(F(T,\varrho)_\infty/L,N)$ vanishes (under the present hypotheses) as a consequence of [11, Lem. 5.4]. Finally, we note that $(\mathrm{H}_4)$ is satisfied if either $p>3$ (obviously) or if $p=3$ is prime to $|G|$ and the modular reductions of $\psi$ and its contragredient $\check{\psi}$ have no common irreducible components over the algebra $\Bbbk[G]$ (which is semisimple in this case).

We can now state our main result on the arithmetic properties of nc-Euler systems. This result constitutes a precise version of Theorem A and concerns the Selmer groups of $T\langle\varrho\rangle$ with respect to the dual $\mathcal{F}^*_{\rm can}$ of the canonical Selmer structure $\mathcal{F}_{\rm can}$ defined by Mazur and Rubin in [33, Def. 3.2.1]. We recall that, in concrete cases, the latter groups recover natural arithmetic objects (see Remark 2.18 below).

**Theorem 2.15.** *Fix a homomorphism $\varrho:\mathcal{O}[[G_K]]\to \mathrm{M}_n(\mathcal{A})$ as in (2.2.1), a pro-$p$ abelian extension $\mathcal{L}$ of $K$ in $K^c$ and a finite extension $F$ of $K$ in $\mathcal{L}$ that satisfy Hypotheses 2.12 and 2.13. Write $L$ for the kernel field $K(\varrho)$ of $\varrho$ and fix a Galois extension $\mathcal{K}$ of $K$ in $K^c$ that contains both $L$ and $\mathcal{L}$. Set $G:=\mathcal{G}_L$ and $E:=LF$ and assume that $T\langle\varrho\rangle$ satisfies Hypothesis 2.8 relative to $\mathcal{K}$. Fix a natural number $r$ and an nc-Euler system $\varepsilon$ in $\mathrm{ES}_r(T,\mathcal{K})$. Then for every subset $\{\varphi_i\}_{1\le i\le r}$ of $\mathrm{Hom}_{\mathcal{O}[\mathcal{G}_E]}(H^1(\mathcal{O}_{E,S(E)},T),\mathcal{O}[\mathcal{G}_E])$ one has*

$$d(\varrho)^{nr}e(\varrho)\cdot(\wedge_{i=1}^{i=r}\varphi_i)(\varepsilon_E)\in \mathrm{Fit}^0_{\mathcal{A}[\mathcal{G}_F]}\big(H^1_{\mathcal{F}^*_{\rm can}}(F,T\langle\varrho\rangle^\vee(1))^\vee\big).$$

*In particular, the displayed product belongs to $\mathcal{A}[\mathcal{G}_F]$ and annihilates the dual Selmer group $H^1_{\mathcal{F}^*_{\rm can}}(F,T\langle\varrho\rangle^\vee(1))^\vee$.*

Theorem 2.15 will be proved in §3.1 and implies explicit restrictions on the Galois structures of the Selmer and Tate-Shafarevich groups of $T$ over the (in general, non-abelian) Galois extension $E$ of $K$.

For example, if $e(\varrho)$ belongs to $\mathcal{O}[G]$ and $d(\varrho)=1$ (as is automatically the case, for example, in the setting of Example 2.2.3(iii)), then one has $\mathcal{A}[\mathcal{G}_F]=e(\varrho)\zeta(\mathcal{O}[\mathcal{G}_E])$ and

$$H^1_{\mathcal{F}^*_{\rm can}}(F,T\langle\varrho\rangle^\vee(1))^\vee=\mathcal{A}[\mathcal{G}_F]^n\otimes_{\mathrm{M}_n(\mathcal{A}[\mathcal{G}_F])}\big(e(\varrho)\cdot H^1_{\mathcal{F}^*_{\rm can}}(E,T^\vee(1))^\vee\big).$$

In this case, therefore, Theorem 2.15 implies that $(\wedge_{i=1}^{i=r}\varphi_i)(e(\varrho)\cdot\varepsilon_E)$ belongs to $\mathcal{O}[\mathcal{G}_E]$ and

$$(\wedge_{i=1}^{i=r}\varphi_i)(e(\varrho)\cdot\varepsilon_E)\cdot H^1_{\mathcal{F}^*_{\rm can}}(E,T^\vee(1))^\vee=(0).$$

To state a more general consequence of Theorem 2.15 we recall (from [38, Def. (8.6.2)]) that, if $N$ is any $\mathcal{O}[[G_K]]$-module that is unramified outside $S(E)$, then the Tate-Shafarevich group $\text{Ш}^2(\mathcal{O}_{E,S(E)},N)$ of $N$ is defined to be the kernel of the natural localisation map

$$H^2(\mathcal{O}_{E,S(E)},N)\xrightarrow{\lambda_{E,S(E),N}}\bigoplus\nolimits_{v\in S(E)}H^2(E_v,N). \tag{2.2.4}$$

For each field $F$ in $\Omega(\mathcal{L}/K)$ we write $\chi_{\varrho,F}:\mathcal{G}_E\to\mathcal{A}_F$ for the $\mathcal{A}_F$-valued character of the action of $\mathcal{G}_E$ on $\mathcal{A}_F^n$ that is induced by $\varrho$, and define an associated 'projector' by setting

$$\mathrm{pr}_{\varrho,F}:=\sum\nolimits_{g\in\mathcal{G}_E}\chi_{\varrho,F}(g)\otimes g\in\mathcal{A}_F[\mathcal{G}_E].$$

For a homomorphism of $\mathcal{Q}$-modules $\epsilon:A_F\to\mathcal{Q}[\mathcal{G}_E]$ we also write $\epsilon_{\mathcal{G}_E}$ for the homomorphism $A_F[\mathcal{G}_E]\to\mathcal{Q}[\mathcal{G}_E]$ that sends each element $\sum_{g\in\mathcal{G}_E}a_gg$ to $\sum_{g\in\mathcal{G}_E}\epsilon(a_g)g$.

The following result will be proved in §3.2.

**Corollary 2.16.** *Assume the notation and hypotheses of Theorem 2.15. Assume also that* $H^0(E, T^\vee(1))$ *vanishes, and write* $m$ *for the lowest common multiple of the orders of the decomposition groups in* $G$ *of places in* $S(E)$*.*

*Then, for any homomorphism of* $\mathcal{O}$*-modules* $\epsilon : \mathcal{A}_F \to \mathcal{O}[\mathcal{G}_E]$*, any subset* $\{\varphi_i\}_{1\le i\le r}$ *of* $\mathrm{Hom}_{\mathcal{O}[\mathcal{G}_E]}(H^1(\mathcal{O}_{E,S(E)}, T), \mathcal{O}[\mathcal{G}_E])$ *and any nc-Euler system* $\varepsilon$ *in* $\mathrm{ES}_r(T, \mathcal{K})$*, the element*

$$d(\varrho)^{nr} m \cdot \epsilon_{\mathcal{G}_E}\big(e(\varrho)\cdot(\wedge_{i=1}^{i=r}\varphi_i)(\varepsilon_E)\cdot \mathrm{pr}_{\varrho,F}\big)$$

*belongs to* $\mathcal{O}[\mathcal{G}_E]$ *and annihilates the dual Tate-Shafarevich group* $\text{Ш}^2(\mathcal{O}_{E,S(E)}, T)^\vee$*.*

**Example 2.17.** Assume the setting of Example 2.11(i), so that $\varrho = \varrho_\psi$ for a character $\psi$ in $\mathrm{Ir}_p(G)$, and take $F = K$ (so that $E = L$). Then $\chi_{\varrho,F} = \psi$, $\mathcal{A}_F = \mathcal{O}$, $e(\varrho) = e_\psi$, $\mathrm{pr}_{\varrho,F} = \mathrm{pr}_\psi := (|G|/\psi(1))e_\psi$, $\mathrm{Nrd}_{\mathbb{Q}_p[G]}(d(\varrho))e_\psi = d(\varrho)^{\psi(1)}e_\psi$ and $d(\varrho)$ is a divisor of $|G|/\psi(1)$. Thus, if $\epsilon$ is the tautological inclusion of $\mathcal{A}_F = \mathcal{O}$ into $\mathcal{O}[G]$, then Corollary 2.16 implies that

$$(|G|/\psi(1))^{\psi(1)r} m \cdot (\wedge_{i=1}^{i=r}\varphi_i)(\varepsilon_E)\cdot \mathrm{pr}_\psi$$

belongs to $\mathcal{O}[G]$ and annihilates $\text{Ш}^2(\mathcal{O}_{L,S(L)}, T)^\vee$.

**Remark 2.18.** Upon appropriate specialization, the Selmer and Tate-Shafarevich groups considered above recover classical modules. To describe two concrete examples (that are respectively discussed in greater detail in [11, §5 and §6]), we fix a field $E$ in $\Omega(\mathcal{K}/K)$.
(i) In the setting of the examples considered in Remark 2.14, the groups $H^1_{\mathcal{F}^*_{\mathrm{can}}}(E, \mathbb{Q}_p/\mathbb{Z}_p)$ and $\text{Ш}^2(\mathcal{O}_{E,S(E)}, \mathbb{Z}_p(1))$ respectively identify with the Pontryagin duals of the $p$-primary subgroups of the ideal class groups of $\mathcal{O}_{E,S_p(E)}$ and $\mathcal{O}_{E,S(E)}$.
(ii) Let $C$ be an elliptic curve defined over $K$ and write $\mathrm{Sel}_p(C/E)$ for its classical $p$-Selmer group over $E$ and $T = T_p(C)$ for its $p$-adic Tate module. The 'strict $p$-Selmer group' $\mathrm{Sel}_p^{\mathrm{str}}(C/E)$ of $C$ over $E$ is defined to be the kernel of the natural localization map

$$\mathrm{Sel}_p(C/E) \to \bigoplus\nolimits_{\mathfrak{p}\in S_p(E)} H^1(E_\mathfrak{p}, T\otimes_{\mathbb{Z}_p}\mathbb{Q}_p/\mathbb{Z}_p).$$

Then one has $H^1_{\mathcal{F}^*_{\mathrm{can}}}(E, T^\vee(1)) = \mathrm{Sel}_p^{\mathrm{str}}(C/E)$ and $\text{Ш}^2(\mathcal{O}_{E,S(E)}, T)$ is the kernel of a natural localisation map

$$\mathrm{Sel}_p^{\mathrm{str}}(C/E) \to \bigoplus\nolimits_{\mathfrak{q}} H^1_{/f}(K_\mathfrak{q}, T_E)^\vee.$$

Here $\mathfrak{q}$ runs over all places in $S(E)\setminus S_p(K)$ and $H^1_{/f}(K_\mathfrak{q}, T_E)$ denotes the image of the natural restriction map $H^1(K_\mathfrak{q}, T_E) \to H^1(K_\mathfrak{q}^{\mathrm{un}}, T_E\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)$, with $K_\mathfrak{q}^{\mathrm{un}}$ the maximal unramified extension of $K_\mathfrak{q}$.

2.2.2. The above results still leave open the key question of whether there exist *any* non-trivial nc-Euler systems. In this section we present an unconditional construction of a family of such systems that can have arbitrarily large rank and finer arithmetic properties than those that are described in Corollary 2.16 above. This construction concerns $p$-adic representations satisfying a range of explicit conditions that are different (and, in general, weaker) than those used in Theorem 2.15 and combines an approach introduced (in a commutative setting) in [12] with the theory of reduced determinant functors from [13, §5].

To state our main result in this regard, for each field $E$ in $\Omega(\mathcal{K}/K)$ we define an idempotent in $\zeta(A[\mathcal{G}_E])$ by setting

$$e_E = e_{E,T} := \sum\nolimits_e e \tag{2.2.5}$$

where in the sum $e$ runs over all primitive central idempotents of $A[\mathcal{G}_E]$ that annihilate the space $H^2(\mathcal{O}_{E,S(E)}, V)$, and we use the $\mathcal{A}[\mathcal{G}_E]$-module

$$Y_E(T^*(1)) := \bigoplus\nolimits_{w\in S_\infty(E)} H^0(E_w, T^*(1)).$$

**Theorem 2.19.** *Assume that $T$ and $\mathcal{K}$ satisfy all of following conditions:*

(a) *Hypothesis 2.8 is satisfied;*
(b) *the $\mathcal{A}$-module $T$ is projective;*
(c) *the $\mathcal{A}$-module $Y_K(T^*(1))$ is free of rank $r = r_T$;*
(d) *all archimedean places of $K$ split completely in $\mathcal{K}$.*

*Then there exists an nc-Euler system $\varepsilon = \varepsilon_{\mathcal{K}/K}(T)$ in $\mathrm{ES}_r(T, \mathcal{K})$ that is canonical up to multiplication by an element of $\xi(\mathcal{A}[[\mathrm{Gal}(\mathcal{K}/K)]])^\times$ and has both of the following properties.*

(i) *The annihilator of $\varepsilon$ in $\xi(\mathcal{A}[[\mathrm{Gal}(\mathcal{K}/K)]])$ is equal to*

$$\xi(\mathcal{A}[[\mathrm{Gal}(\mathcal{K}/K)]]) \cap \prod\nolimits_{L\in\Omega(\mathcal{K}/K)} \zeta(A[\mathcal{G}_L])(1 - e_L).$$

(ii) *For each $L$ in $\Omega(\mathcal{K}/K)$ one has*

$$\big(\wedge_{i=1}^{i=r}\varphi_i\big)(\varepsilon_L) \in \mathrm{Fit}^0_{\mathcal{A}[\mathcal{G}_L]}\big(H^2(\mathcal{O}_{L,S(L)}, T)\big)$$

*for every subset $\{\varphi_i\}_{1\le i\le r}$ of $\mathrm{Hom}_{\mathcal{A}[\mathcal{G}_L]}(H^1(\mathcal{O}_{L,S(L)}, T), \mathcal{A}[\mathcal{G}_L])$.*

**Remark 2.20.** The proof of claim (ii) of Theorem 2.19 that is given in §4.3 will establish (in the equality (4.3.2)) a more precise result in which elements of the form $\big(\wedge_{i=1}^{i=r}\varphi_i\big)(\varepsilon_L)$ determine the $r$-th Fitting invariant of a presentation of the module $H^2(\mathcal{O}_{L,S(L)}, T)\oplus\mathcal{A}[\mathcal{G}_L]^r$

We next describe some Iwasawa-theoretic properties of the Euler systems constructed in Theorem 2.19.

To do this we fix a $p$-adic analytic extension $\mathcal{L}$ of $K$ in $\mathcal{K}$ and use the Iwasawa algebra

$$\mathcal{A}[[\mathcal{L}/K]] := \varprojlim\nolimits_{L\in\Omega(\mathcal{L}/K)} \mathcal{A}[\mathcal{G}_L]$$

where the transition morphisms are the natural projection maps.

We recall that this ring is both left and right noetherian (by [32, V, 2.2.4]) and has a total quotient ring that we denote by $Q(\mathcal{A}[[\mathcal{L}/K]])$. In fact, if $\mathrm{Gal}(\mathcal{L}/K)$ is a torsion-free prop-$p$ group, then $\mathcal{O}[[\mathcal{L}/K]]$ has no proper zero-divisors (by [39]) and so $Q(\mathcal{O}[[\mathcal{L}/K]])$ is a skew field (see [26]).

In the general case, there exists a field $L_0$ in $\Omega(\mathcal{L}/K)$ so that $\mathrm{Gal}(\mathcal{L}/L_0)$ is a torsion-free pro-$p$ group. One then has $S(L) = S(L_0)$ for all $L$ in $\Omega(\mathcal{L}/L_0)$ and in each degree $i$ we set

$$H^i(\mathcal{O}_{\mathcal{L}}, T) := \varprojlim\nolimits_{L\in\Omega(\mathcal{L}/L_0)} H^i(\mathcal{O}_{L,S(L)}, T)$$

where the limit is taken with respect to the natural corestriction maps.

Taking account of Lemma 2.10(ii), we also define

$$\bigcap\nolimits^r_{\mathcal{A}[[\mathcal{L}/K]]} H^1(\mathcal{O}_{\mathcal{L}}, T) := \varprojlim\nolimits_{L\in\Omega(\mathcal{L}/L_0)} \bigcap\nolimits^r_{\mathcal{A}[\mathcal{G}_L]} H^1(\mathcal{O}_{L,S(L)}, T)$$

where the transition morphisms are induced by corestriction.

Finally, we set

$$\text{(2.2.6)}\quad \mathrm{Hom}^*_{\mathcal{A}[[\mathcal{L}/K]]}\big(H^1(\mathcal{O}_{\mathcal{L}}, T), \mathcal{A}[[\mathcal{L}/K]]\big) := \varprojlim_{L\in\Omega(\mathcal{L}/L_0)} \mathrm{Hom}_{\mathcal{A}[\mathcal{G}_L]}\big(H^1(\mathcal{O}_{L,S(L)}, T), \mathcal{A}[\mathcal{G}_L]\big)$$

where the limit is taken with respect to the maps $\varrho^*_{L'/L}$ that occur in diagram (2.1.3) with $F'/F$ replaced by $L'/L$ for $L_0 \subseteq L \subseteq L' \subset \mathcal{L}$.

Then for any subset $\{\varphi_i\}_{1\le i\le r}$ of $\mathrm{Hom}^*_{\mathcal{A}[[\mathcal{L}/K]]}\big(H^1(\mathcal{O}_{\mathcal{L}}, T), \mathcal{A}[[\mathcal{L}/K]]\big)$ and any element $\eta = (\eta_L)_L$ of $\bigcap^r_{\mathcal{A}[[\mathcal{L}/K]]} H^1(\mathcal{O}_{\mathcal{L}}, T)$ the commutativity of (2.1.4) implies that we obtain a well-defined element of the limit $\varprojlim_{L\in\Omega(\mathcal{L}/L_0)} \zeta(A[\mathcal{G}_L])$, where the transition morphisms are the natural projection maps, by setting

$$\big(\wedge^{i=r}_{i=1}\varphi_i\big)(\eta) := \big(\big(\wedge^{i=r}_{i=1}\varphi_{i,L}\big)(\eta_L)\big)_L$$

where $\varphi_{i,L}$ is the projection of $\varphi_i$ to $\mathrm{Hom}_{\mathcal{A}[\mathcal{G}_L]}\big(H^1(\mathcal{O}_{L,S(L)}, T), \mathcal{A}[\mathcal{G}_L]\big)$.

**Remark 2.21.** It is easily seen that all of the definitions made above are independent of the choice of the field $L_0$. In addition, Lemma 4.14 below gives an explicit description of the image of the natural homomorphism

$$\mathrm{Hom}^*_{\mathcal{A}[[\mathcal{L}/K]]}\big(H^1(\mathcal{O}_{\mathcal{L}}, T), \mathcal{A}[[\mathcal{L}/K]]\big) \to \mathrm{Hom}_{\mathcal{A}[[\mathcal{L}/K]]}\big(H^1(\mathcal{O}_{\mathcal{L}}, T), \mathcal{A}[[\mathcal{L}/K]]\big).$$

In the following result we say that an $\mathcal{A}[[\mathcal{L}/K]]$-module is 'central torsion' if it is annihilated by a non-zero divisor of $\zeta(\mathcal{A}[[\mathcal{L}/K]])$.

We will also write $\delta(\mathcal{A}[[\mathcal{L}/K]])$ for the ideal of $\zeta(\mathcal{A}[[\mathcal{L}/K]])$ that is given by the limit $\varprojlim_L \delta(\mathcal{A}[\mathcal{G}_L])$ as $L$ runs over $\Omega(\mathcal{L}/K)$ and the transition morphisms are induced by [13, Lem. 3.7(vii)] and the natural projection maps $\mathcal{A}[\mathcal{G}_{L'}] \to \mathcal{A}[\mathcal{G}_L]$ for $L \subseteq L'$.

**Theorem 2.22.** *Fix a $p$-adic analytic extension $\mathcal{L}$ of $K$ in $\mathcal{K}$. Then the nc-Euler system $\varepsilon$ constructed (under the stated hypotheses) in Theorem 2.19 has all of the following properties.*

(i) *The element $\varepsilon_{\mathcal{L}} := \big(\varepsilon_L\big)_{L\in\Omega(\mathcal{L}/L_0)}$ belongs to $\bigcap^r_{\mathcal{A}[[\mathcal{L}/K]]} H^1(\mathcal{O}_{\mathcal{L}}, T)$.*

*In the remainder of the result we assume $\mathcal{A} = \mathcal{O}$ and set $R_{\mathcal{L}} := \mathcal{O}[[\mathcal{L}/K]]$.*

(ii) *Assume that* $\mathrm{Gal}(\mathcal{L}/K)$ *has rank one. Then the $R_{\mathcal{L}}$-module $H^2(\mathcal{O}_{\mathcal{L}}, T)$ is a torsion module if and only if there exists a subset $\{\varphi_i\}_{1\le i\le r}$ of* $\mathrm{Hom}^*_{R_{\mathcal{L}}}\big(H^1(\mathcal{O}_{\mathcal{L}}, T), R_{\mathcal{L}}\big)$ *for which $\big(\wedge^{i=r}_{i=1}\varphi_i\big)(\varepsilon_{\mathcal{L}})$ is a unit of $Q(R_{\mathcal{L}})$.*

(iii) *Assume that* $\mathrm{Gal}(\mathcal{L}/K)$ *has rank at least two. Then the $R_{\mathcal{L}}$-module $H^2(\mathcal{O}_{\mathcal{L}}, T)$ is a central torsion module if there exists a subset $\{\varphi_i\}_{1\le i\le r}$ of* $\mathrm{Hom}^*_{R_{\mathcal{L}}}\big(H^1(\mathcal{O}_{\mathcal{L}}, T), R_{\mathcal{L}}\big)$ *for which $\big(\wedge^{i=r}_{i=1}\varphi_i\big)(\varepsilon_{\mathcal{L}})$ is a non-zero divisor in the ideal $\delta(R_{\mathcal{L}})$ of $\zeta(R_{\mathcal{L}})$.*

(iv) *If $\mathcal{L}/K$ is abelian, then the following conditions are equivalent:*

(a) *$\varepsilon_{\mathcal{L}}$ has the property stated in (ii), respectively (iii).*

(b) *$\varepsilon_{\mathcal{L}}$ is a generator of the $Q(R_{\mathcal{L}})$-module generated by $\bigwedge^r_{R_{\mathcal{L}}} H^1(\mathcal{O}_{\mathcal{L}}, T)$.*

(c) *$\varepsilon_{\mathcal{L}}$ is not annihilated by any non-zero divisor of $R_{\mathcal{L}}$.*

The proof of this result is given in §4.4.

## 3. Non-commutative Euler systems and Selmer groups

In this section we deduce the results of Theorem 2.15 and Corollary 2.16 from the theory of higher rank Euler and Kolyvagin systems developed by Sakamoto and the present authors.

Throughout the section, we assume the hypotheses and notation of Theorem 2.15.

3.1. **The proof of Theorem 2.15.** We set $L := K(\varrho)$ and $G := \mathcal{G}_L$ and, since $L$ is disjoint from $\mathcal{L}$, for each $F$ in $\Omega(\mathcal{L}/K)$ we use the identification

$$\mathcal{Q}[\mathcal{G}_{LF}] \cong \mathcal{Q}[G] \otimes_{\mathcal{Q}} \mathcal{Q}[\mathcal{G}_F]$$

to regard $e := e(\varrho)$ as an idempotent of $\mathcal{Q}[\mathcal{G}_{LF}]$. We also set $\mathcal{A}_F := \mathcal{A} \otimes_{\mathcal{O}} \mathcal{O}[\mathcal{G}_F]$ and $A_F = \mathcal{Q} \cdot \mathcal{A}_F$ and consider the $\mathcal{O}$-order

$$\mathcal{A}(\varrho)_F := \mathcal{A}(\varrho) \otimes_{\mathcal{O}} \mathcal{O}[\mathcal{G}_F] \cong \mathrm{M}_n(\mathcal{A}_F) \oplus \mathcal{O}[\mathcal{G}_{LF}](1-e)$$

in $\mathcal{Q}[\mathcal{G}_{LF}]$.

We start by proving a useful general result.

**Lemma 3.1.** *Set* $\Gamma := \mathcal{G}_{LF}$.

(i) *For each* $\mathcal{O}[\Gamma]$*-lattice* $X$ *the following claims are valid.*

(a) *There are natural identifications*

$$\begin{aligned} e{\bigwedge}^r_{\mathcal{Q}[\Gamma]}\mathcal{Q}\cdot X &= {\bigwedge}^r_{\mathrm{M}_n(A_F)} e(\mathcal{Q}\cdot X) \\ &= {\bigwedge}^{nr}_{A_F}\left(A_F^n \otimes_{\mathrm{M}_n(A_F)} e(\mathcal{Q}\cdot X)\right) = {\bigwedge}^{nr}_{A_F}\mathcal{Q}\cdot\left(\mathcal{A}_F^n\otimes_{\mathcal{O}} X\right)^{\Gamma}. \end{aligned}$$

(b) *The identification* $(\mathcal{A}_F^n \otimes_{\mathcal{O}} \mathcal{O}[\Gamma])^\Gamma \cong \mathcal{A}_F^n$ *induces a map*

$$\begin{aligned} \pi_{X,e} : \mathrm{Hom}_{\mathcal{O}[\Gamma]}(X, \mathcal{O}[\Gamma]) &\to \mathrm{Hom}_{\mathcal{A}_F}\left((\mathcal{A}_F^n \otimes_{\mathcal{O}} X)^\Gamma, \mathcal{A}_F\right)^n, \\ f &\mapsto \mathrm{id} \otimes_{\mathcal{O}} f, \end{aligned}$$

*the cokernel of which is annihilated by* $d(\varrho)$.

(c) *With respect to the identification in claim (a), there is an inclusion*

$$d(\varrho)^{nr} e \cdot {\bigcap}^r_{\mathcal{O}[\Gamma]} X \subseteq {\bigcap}^{nr}_{\mathcal{A}_F}\left(\mathcal{A}_F^n \otimes_{\mathcal{O}} X\right)^{\Gamma}.$$

(ii) *Let* $C$ *be an object of* $\mathsf{D}^{\mathrm{perf}}(\mathcal{O}[\Gamma])$ *that is acyclic in degrees less than one and such that* $H^1(C)$ *is* $\mathcal{O}$*-torsion-free. Then there is a natural identification of* $\mathcal{A}_F$*-modules*

$$H^1\left(\mathcal{A}_F^n \otimes^{\mathsf{L}}_{\mathcal{O}[\Gamma]} C\right) = \left(\mathcal{A}_F^n \otimes_{\mathcal{O}} H^1(C)\right)^{\Gamma}.$$

*Proof.* The displayed identifications in claim (i)(a) respectively follow from the equality $e \cdot \mathcal{Q}[\Gamma] = \mathrm{M}_n(A_F)$, the general result of [13, Th. 4.19(vii)] and the canonical identification $A_F^n \otimes_{\mathrm{M}_n(A_F)} e(\mathcal{Q}\cdot X) = \mathcal{Q}\cdot\left(\mathcal{A}_F^n \otimes_{\mathcal{O}} X\right)^{\Gamma}$.

Since $\mathrm{M}_n(\mathcal{A}_F)$ is a direct factor of the algebra $\mathcal{A}(\varrho)_F$, to prove claim (i)(b) it is enough to show that the cokernel of the analogous map

$$\mathrm{Hom}_{\mathcal{O}[\Gamma]}(X, \mathcal{O}[\Gamma]) \to N := \mathrm{Hom}_{\mathcal{A}(\varrho)_F}\left((\mathcal{A}(\varrho)_F \otimes_{\mathcal{O}} X)^{\Gamma}, \mathcal{A}(\varrho)_F\right)$$

is annihilated by $d(\varrho)$. In addition, since $X$ identifies with the submodule $(\mathcal{O}[\Gamma]\otimes_{\mathcal{O}} X)^{\Gamma}$ of $(\mathcal{A}(\varrho)_F\otimes_{\mathcal{O}} X)^{\Gamma}$, for any $\theta$ in $N$ one has $\theta(X)\in\mathcal{A}(\varrho)_F$. It is therefore enough to note that $d(\varrho)$ annihilates the quotient group $\mathcal{A}(\varrho)_F/\mathcal{O}[\Gamma]\cong\big(\mathcal{A}(\varrho)/\mathcal{O}[G]\big)\otimes_{\mathcal{O}}\mathcal{O}[\mathcal{G}_F]$.

We next fix an element $x$ of $\bigcap_{\mathcal{O}[\Gamma]}^{r}X$ and use claim (i)(a) to regard $e\cdot x$ as an element of $\bigwedge_{A_F}^{nr}\big(A_F^n\otimes_{\mathcal{O}} X\big)^{\Gamma}$. Then, to prove claim (i)(c), we need to show that

$$d(\varrho)^{nr}\cdot(\wedge_{i=0}^{i=nr-1}\theta_i)\big(e\cdot x\big)\in\mathcal{A}_F$$

for every subset $\{\theta_i\}_{0\le i<nr}$ of $\mathrm{Hom}_{\mathcal{A}_F}\big((\mathcal{A}_F^n\otimes_{\mathcal{O}} X)^{\Gamma},\mathcal{A}_F\big)$. To show this we use claim (i)(b) to fix, for each $j$ with $0\le j<r$, a map $\varphi_j$ in $\mathrm{Hom}_{\mathcal{O}[\Gamma]}(X,\mathcal{O}[\Gamma])$ such that

$$\pi_{X,e}(\varphi_j)=d(\varrho)\cdot(\theta_{jn},\theta_{jn+1},\cdots,\theta_{jn+n-1}).$$

Then one has

$$\begin{aligned}d(\varrho)^{nr}\cdot(\wedge_{i=0}^{i=nr-1}\theta_i)\big(e\cdot x\big)&=\big(\wedge_{j=0}^{j=r-1}(\wedge_{i=0}^{i=n-1}d(\varrho)\theta_{jn+i})\big)(e\cdot x)\\&=e\cdot(\wedge_{j=0}^{j=r-1}\varphi_j)(x)\in\ \xi(\mathcal{O}[\Gamma])e.\end{aligned}$$

To prove claim (i)(c) it is then enough to note that $\xi(\mathcal{O}[\Gamma])e$ is contained in $\xi(\mathcal{A}(\varrho)_F)e=\xi(\mathrm{M}_n(\mathcal{A}_F))$ which, by the first assertion of [13, Th. 4.19(vii)], is equal to $\mathcal{A}_F$.

Turning to claim (ii) we recall that any torsion-free finitely generated $\mathcal{O}[\Gamma]$-module that has finite projective dimension is necessarily projective (cf. [1, Th. 8]). Given the conditions on $C$, we can therefore use a general construction of homological algebra (as, for example, in [20, Rapport, Lem. 4.7]) to show that $C$ is isomorphic in $\mathsf{D}^{\mathrm{perf}}(\mathcal{O}[\Gamma])$ to a bounded complex $P^\bullet$ of finitely generated projective $\mathcal{O}[\Gamma]$-modules that has $P^i=0$ for all $i<1$.

It follows that $\mathcal{A}_F^n\otimes^{\mathsf{L}}_{\mathcal{O}[\Gamma]}C$ is isomorphic in $\mathsf{D}(\mathcal{A}_F)$ to $\mathcal{A}_F^n\otimes_{\mathcal{O}[\Gamma]}P^\bullet$, and hence that $H^1(\mathcal{A}_F^n\otimes^{\mathsf{L}}_{\mathcal{O}[\Gamma]}C)$ is equal to $\ker(\mathrm{id}\otimes_{\mathcal{O}[\Gamma]}d)$, with 'id' the identity map on $\mathcal{A}_F^n$ and $d$ the differential of $P^\bullet$ in degree one. We can therefore deduce the claimed result from the exact commutative diagram

$$\begin{tikzcd}0 \arrow[r] & \big(\mathcal{A}_F^n\otimes_{\mathcal{O}}H^1(C)\big)^{\Gamma} \arrow[r, "\iota^J"] & \big(\mathcal{A}_F^n\otimes_{\mathcal{O}}P^1\big)^{\Gamma} \arrow[r, "\mathrm{id}\otimes_{\mathcal{O}}d"] & (\mathcal{A}_F^n\otimes_{\mathcal{O}}P^2)^{\Gamma}\\ & & \mathcal{A}_F^n\otimes_{\mathcal{O}[\Gamma]}P^1 \arrow[u, "\cong"] \arrow[r, "\mathrm{id}\otimes_{\mathcal{O}[\Gamma]}d"] & \mathcal{A}_F^n\otimes_{\mathcal{O}[\Gamma]}P^2. \arrow[u, "\cong"']\end{tikzcd}$$

Here the two vertical arrows are induced by the maps $\mathcal{A}_F^n\otimes_{\mathcal{O}}P^i\to(\mathcal{A}_F^n\otimes_{\mathcal{O}}P^i)^{\Gamma}$ sending each element $a\otimes x$ to $\sum_{\gamma\in\Gamma}a\gamma^{-1}\otimes\gamma(x)$ and are bijective since $P^i$ is a projective $\mathcal{O}[\Gamma]$-module. In addition, the upper row of the diagram is exact since $\mathcal{A}_F^n$ is a flat $\mathcal{O}$-module and so the functor $Y\mapsto(\mathcal{A}_F^n\otimes_{\mathcal{O}}Y)^{\Gamma}$ is left exact (from the category of left $\mathcal{O}[\Gamma]$-modules to the category of left $\mathcal{A}_F$-modules). □

We can now establish a concrete link between the theories of non-commutative and commutative Euler systems that is key to the proof of Theorem 2.15.

**Proposition 3.2.** *For any system $\varepsilon$ in $\mathrm{ES}_r(T,L\mathcal{L})$, the family*

$$\varepsilon\langle\varrho\rangle:=\{d(\varrho)^{nr}e(\varrho)\cdot\varepsilon_{LF'}\}_{F'\in\Omega(\mathcal{L}/K)}$$

*defines an element of $\mathrm{ES}_{nr}(T\langle\varrho\rangle,\mathcal{L})$.*

*Proof.* Fix fields $F_1$ and $F_2$ in $\Omega(\mathcal{L}/K)$ and, for $i = 1, 2$, set $\mathcal{A}_i := \mathcal{A}_{F_i}$, $A_i := A_{F_i}$, $G_i := \mathcal{G}_{F_i}$, $E_i := LF_i$, $\Gamma_i := \mathcal{G}_{E_i} \cong G \times G_i$, $\mathcal{R}_i := \mathcal{O}[\Gamma_i]$ and $T_i := \mathcal{R}_i \otimes_{\mathcal{O}} T$.

We claim first that there are canonical identifications of $\mathcal{A}_i$-modules

$$H^1(\mathcal{O}_{F_i,S(F_i)}, T\langle\varrho\rangle) \cong \left(\mathcal{A}_i^n \otimes_{\mathcal{O}} H^1(\mathcal{O}_{E_i,S(E_i)}, T)\right)^{\Gamma_i}. \tag{3.1.1}$$

To prove this we note that the natural isomorphism of $\mathcal{A}_i[[G_K]]$-modules

$$\mathcal{O}[G_i] \otimes_{\mathcal{O}} T\langle\varrho\rangle \cong \mathcal{O}[G_i] \otimes_{\mathcal{O}} \left(\mathcal{A}^n \otimes_{\mathcal{O}} T\right) \cong \mathcal{A}_i^n \otimes_{\mathcal{O}} T \cong \mathcal{A}_i^n \otimes_{\mathcal{R}_i} T_i$$

combines with Shapiro's Lemma to give an identification

$$H^1(\mathcal{O}_{F_i,S(F_i)}, T\langle\varrho\rangle) \cong H^1(\mathcal{O}_{K,S(F_i)}, \mathcal{O}[G_i] \otimes_{\mathcal{O}} T\langle\varrho\rangle) \cong H^1(\mathcal{O}_{K,S(F_i)}, \mathcal{A}_i^n \otimes_{\mathcal{R}_i} T_i).$$

In addition, since $T_i$ is a projective $\mathcal{R}_i$-module, the complex $\mathsf{R}\Gamma(\mathcal{O}_{K,S(F_i)}, T_i)$ belongs to $\mathsf{D}^{\rm perf}(\mathcal{R}_i)$ (by Flach [24, Th. 5.1]) and there is a natural isomorphism in $\mathsf{D}(\mathcal{A}_i)$ of the form

$$\mathsf{R}\Gamma(\mathcal{O}_{K,S(F_i)}, \mathcal{A}_i^n \otimes_{\mathcal{R}_i} T_i) \cong \mathcal{A}_i^n \otimes^{\mathsf{L}}_{\mathcal{R}_i} \mathsf{R}\Gamma(\mathcal{O}_{K,S(F_i)}, T_i) \cong \mathcal{A}_i^n \otimes^{\mathsf{L}}_{\mathcal{R}_i} \mathsf{R}\Gamma(\mathcal{O}_{E_i,S(F_i)}, T) \tag{3.1.2}$$

(by Fukaya and Kato [25, Prop. 1.6.5]).

Thus, since Hypothesis 2.8 implies that the $\mathcal{O}$-module $H^j(\mathcal{O}_{E_i,S(F_i)}, T)$ vanishes for $j = 0$ and is torsion-free for $j = 1$, the claimed isomorphism (3.1.1) is obtained as a consequence of Lemma 3.1(ii) with $F$, $\Gamma$ and $C$ taken to be $F_i$, $\mathsf{R}\Gamma(\mathcal{O}_{E_i,S(F_i)}, T)$ and $\Gamma_i$ respectively.

Next we note that the definition of $\mathrm{ES}_r(T, L\mathcal{L})$ implies directly that each element $\varepsilon_{E_i}$ belongs to $\bigcap^r_{\mathcal{O}[\Gamma_i]} H^1(\mathcal{O}_{E_i,S(F_i)}, T)$. Upon combining the identification (3.1.1) with the general result of Lemma 3.1(i)(c), we can therefore deduce that there is a containment

$$\varepsilon\langle\varrho\rangle_{F_i} = d(\varrho)^{nr} e(\varrho) \cdot \varepsilon_{E_i} \in \bigcap\nolimits^{nr}_{\mathcal{A}_i} H^1(\mathcal{O}_{F_i,S(F_i)}, T\langle\varrho\rangle)$$

for both $i = 1, 2$.

To prove the claimed result, it is therefore enough to show that if $F_1 \subseteq F_2$, then in $\bigwedge^{nr}_{A_1} H^1(\mathcal{O}_{E_1,S(F_1)}, V\langle\varrho\rangle)$ one has

$$\mathrm{Cor}^{nr}_{F_2/F_1}(\varepsilon\langle\varrho\rangle_{F_2}) = \left(\prod\nolimits_{v \in S(F_2)\setminus S(F_1)} \det_{A_1}(1 - \sigma_v \mid V\langle\varrho\rangle^*_{F_1}(1))^{\#}\right)(\varepsilon\langle\varrho\rangle_{F_1})$$

To check this, we recall that $L$ is disjoint from $F_2$ (by condition ($\mathrm{E}_4$) in Hypothesis 2.12), and hence that, with respect to the respective identifications

$$\bigwedge\nolimits^{nr}_{A_i} H^1(\mathcal{O}_{F_i,S(F_i)}, V\langle\varrho\rangle) = e(\varrho) \cdot \bigwedge\nolimits^{r}_{\mathrm{M}_n(A_i)} e \cdot H^1(\mathcal{O}_{E_i,S(F_i)}, V),$$

the map $\mathrm{Cor}^{nr}_{F_2/F_1}$ can be computed as the $e(\varrho)$-component of the corestriction map $\mathrm{Cor}^r_{E_2/E_1}$ that arises in the definition of $\mathrm{ES}_r(T, L\mathcal{L})$.

In particular, as $\varepsilon \in \mathrm{ES}_r(T, L\mathcal{L})$, to prove the required equality it is enough to show that

$$\begin{aligned} e(\varrho) \cdot \prod\nolimits_{v \in S(E_2)\setminus S(E_1)} &\mathrm{Nrd}_{\mathcal{Q}[G_1]}(1 - \sigma_v \mid V^*(1)_{E_1})^{\#} \\ &= \prod\nolimits_{v \in S(F_2)\setminus S(F_1)} \det_{A_1}(1 - \sigma_v \mid V\langle\varrho\rangle^*(1)_{F_1})^{\#}. \end{aligned}$$

To see this we note first that any place of $K$ that ramifies in the kernel field $L = K(\varrho)$ belongs to $S_{\mathrm{ram}}(T\langle\varrho\rangle)$ and hence that $S(E_2) \setminus S(E_1) = S(F_2) \setminus S(F_1)$. It is then enough to note that for each place $v$ in the latter set, the definition of reduced norm implies that

$$\begin{aligned} e(\varrho)\cdot \mathrm{Nrd}_{\mathcal{Q}[G_1]}(1-\sigma_v \mid V^*(1)_{E_1}) &= \mathrm{Nrd}_{\mathrm{M}_n(A_1)}(1-\sigma_v \mid \mathrm{M}_n(A_1)\otimes_{\mathcal{Q}} V^*(1))\\ &= \mathrm{det}_{A_1}(1-\sigma_v \mid A_1^n \otimes_{\mathcal{Q}} V^*(1))\\ &= \mathrm{det}_{A_1}(1-\sigma_v \mid V\langle\varrho\rangle^*(1)). \end{aligned}$$

□

To proceed, we next note that Lemma 3.1(ii) combines with the isomorphism (3.1.1) and the explicit definition of reduced exterior powers to imply that for any (ordered) set of maps

$$\{\varphi_i\}_{1\le i\le r} \subset \mathrm{Hom}_{\mathcal{O}[\mathcal{G}_{LF}]}(H^1(\mathcal{O}_{LF,S(LF)},T),\mathcal{O}[\mathcal{G}_{LF}]),$$

there exists a corresponding set of maps

$$\{\theta_i\}_{1\le i\le nr} \subset \mathrm{Hom}_{\mathcal{A}_F}\left(H^1(\mathcal{O}_{F,S(F)},T\langle\varrho\rangle),\mathcal{A}_F\right)$$

for which one has

$$d(\varrho)^{nr}e(\varrho)\cdot(\wedge_{i=1}^{i=r}\varphi_i)(\varepsilon_{LF}) = (\wedge_{i=1}^{i=nr}\theta_i)(\varepsilon\langle\varrho\rangle_F),$$

where the element $\varepsilon\langle\varrho\rangle_F$ is as defined in Proposition 3.2.

By combining the latter observation with the main result of the theory of higher rank (commutative) Euler systems we can then deduce (via a direct application of [11, Th. 3.6(iii)(c)]) that, under the hypotheses of Theorem 2.15, there is a containment

$$d(\varrho)^{nr}e(\varrho)\cdot(\wedge_{i=1}^{i=r}\varphi_i)(\varepsilon_{LF}) \in \mathrm{Fit}^0_{\mathcal{A}_F}\left(H^1_{\mathcal{F}^*_{\mathrm{can}}}(F,T\langle\varrho\rangle^\vee(1))^\vee\right).$$

By a classical property of commutative Fitting ideals (that can also be seen by combining claims (iii) and (vii) of [13, Th. 3.20]), we can therefore conclude that the element

$$d(\varrho)^{nr}e(\varrho)\cdot(\wedge_{i=1}^{i=r}\varphi_i)(\varepsilon_{LF})$$

belongs to $\mathcal{A}_F$ and annihilates $H^1_{\mathcal{F}^*_{\mathrm{can}}}(F,T\langle\varrho\rangle^\vee(1))^\vee$. This proves Theorem 2.15.

**Remark 3.3.** The above argument relies on a reduction (via Morita equivalence) to the main result of [11]. It is perhaps possible that, by incorporating into the approach of [11] a reasonable theory of (higher rank) Kolyvagin systems for representations over non-commutative zero-dimensional Gorenstein rings, one could prove a version of Theorem 2.15 that gives finer information on Galois structure.

3.2. **The proof of Corollary 2.16.** At the outset, we recall from [11, Exam. 2.7(iii)] that the Poitou-Tate global duality theorem identifies $Ш^2(\mathcal{O}_{F,S(E)},T\langle\varrho\rangle)$ with the dual Selmer group $H^1_{\mathcal{F}^*_{\mathrm{rel}}}(F,T\langle\varrho\rangle^\vee(1))^\vee$ of $T\langle\varrho\rangle$ with respect to the dual of the 'relaxed' Selmer structure $\mathcal{F}_{\mathrm{rel}}$ defined in [11, Exam. 2.4]. From the general result of [10, Th. 3.1], we can therefore deduce that the $\mathcal{A}_F$-module $Ш^2(\mathcal{O}_{F,S(E)},T\langle\varrho\rangle)$ is isomorphic to a submodule of $H^1_{\mathcal{F}^*_{\mathrm{can}}}(F,T\langle\varrho\rangle^\vee(1))^\vee$.

Hence, if $\varepsilon \in \mathrm{ES}_r(T,L\mathcal{L})$, then Theorem 2.15 implies that for every subset $\{\varphi_i\}_{1\le i\le r}$ of $\mathrm{Hom}_{\mathcal{O}[\mathcal{G}_E]}(H^1(\mathcal{O}_{E,S(E)},T),\mathcal{O}[\mathcal{G}_E])$ the element $d(\varrho)^{nr}e(\varrho)\cdot(\wedge_{i=1}^{i=r}\varphi_i)(\varepsilon_E)$ belongs to $\mathcal{A}_F$ and annihilates $Ш^2(\mathcal{O}_{F,S(E)},T\langle\varrho\rangle)$. Given this fact, Corollary 2.16 follows directly from claim (ii) of the following result.

**Proposition 3.4.** *Fix $F$ in $\Omega(\mathcal{L}/K)$, set $E := LF$ and assume $H^0(E, T^\vee(1))$ vanishes. Then the following claims are valid.*

(i) *There exists a natural exact sequence of $\mathcal{A}_F$-modules*

$$\mathrm{Tor}_1^{\mathcal{O}[\mathcal{G}_E]}\big(\mathcal{A}_F^n, \bigoplus\nolimits_{v\in S(E)} H^2(E_v, T)\big) \to \mathcal{A}_F^n \otimes_{\mathcal{O}[\mathcal{G}_E]} Ш^2(\mathcal{O}_{E,S(E)}, T) \to Ш^2(\mathcal{O}_{F,S(E)}, T\langle\varrho\rangle) \to 0,$$

*where, for each $v \in S(E)$, we write $H^2(E_v, T)$ for the direct sum of $H^2(E_w, T)$ over all places of $E$ that lie above $v$.*

(ii) *Fix an element $a$ of $\mathcal{A}_F$ that annihilates $Ш^2(\mathcal{O}_{F,S(E)}, T\langle\varrho\rangle)$ and a map of $\mathcal{O}$-modules $\epsilon : \mathcal{A}_F \to \mathcal{O}[\mathcal{G}_E]$. Then, with $m$ denoting the lowest common multiple of the orders of the decomposition groups in $\mathcal{G}_E$ of places in $S(E)$, the element $\epsilon_{\mathcal{G}_E}\big(ma \cdot \mathrm{pr}_{\varrho,F}\big)$ belongs to $\mathcal{O}[\mathcal{G}_E]$ and annihilates $Ш^2(\mathcal{O}_{E,S(E)}, T)^\vee$.*

*Proof.* Set $Ш := Ш^2(\mathcal{O}_{E,S(E)}, T)$ and $G := \mathcal{G}_E$.

Then, since $H^0(E, T^\vee(1))$ vanishes, Poitou-Tate global duality implies that the localisation map $\lambda_{E,S(E),T}$ in (2.2.4) (with $N = T$) is surjective, and hence that there exists a tautological short exact sequence of $\mathcal{O}[G]$-modules

$$0 \to Ш \to H^2(\mathcal{O}_{E,S(E)}, T) \to \bigoplus\nolimits_{v\in S(E)} H^2(E_v, T) \to 0.$$

In addition, since the complex $\mathsf{R}\Gamma(\mathcal{O}_{E,S(E)}, T)$ is acyclic in degrees greater than two (as $p$ is odd), the isomorphism (3.1.2) induces an isomorphism of $\mathcal{A}_F$-modules

$$\mathcal{A}_F^n \otimes_{\mathcal{O}[G]} H^2(\mathcal{O}_{E,S(E)}, T) \cong H^2(\mathcal{O}_{F,S(E)}, T\langle\varrho\rangle).$$

By a similar argument, with $\mathsf{R}\Gamma(\mathcal{O}_{E,S(E)}, T)$ replaced by the direct sum of $\mathsf{R}\Gamma(E_w, T)$ over all places $w$ of $E$ that lie above a given place $v$ in $S(E)$, there is also a natural isomorphism of $\mathcal{A}_F$-modules

$$\mathcal{A}_F^n \otimes_{\mathcal{O}[G]} H^2(E_v, T) \cong H^2(E_v, T\langle\varrho\rangle).$$

Given these isomorphisms, the exact sequence in claim (i) is obtained by applying the functor $\mathcal{A}_F^n \otimes_{\mathcal{O}[G]} -$ to the tautological exact sequence given above.

In order to prove claim (ii), we note that the $\mathcal{O}$-module $\mathcal{A}_F^n$ is torsion-free and hence that, setting $X := \bigoplus_{v\in S(E)} H^2(E_v, T)$, the universal coefficient spectral sequence

$$\mathrm{Tor}_a^{\mathcal{O}[G]}\big(\mathrm{Tor}_b^{\mathcal{O}}(\mathcal{A}_F^n, X), \mathcal{O}\big) \Longrightarrow \mathrm{Tor}_{a+b}^{\mathcal{O}[G]}\big(\mathcal{A}_F^n, X\big)$$

collapses to give an isomorphism

$$\begin{aligned}\mathrm{Tor}_1^{\mathcal{O}[G]}\big(\mathcal{A}_F^n, X\big) &\cong \bigoplus\nolimits_{v\in S(E)} \mathrm{Tor}_1^{\mathcal{O}[G]}\big(\mathcal{A}_F^n \otimes_{\mathcal{O}} H^2(E_v, T), \mathcal{O}\big)\\ &\cong \bigoplus\nolimits_{v\in S(E)} H_1(G, \mathcal{A}_F^n \otimes_{\mathcal{O}} H^2(E_v, T)\big)\\ &\cong \bigoplus\nolimits_{v\in S(E)} H_1(G_w, \mathcal{A}_F^n \otimes_{\mathcal{O}} H^2(E_w, T)\big).\end{aligned}$$

Here, for each $v$ in $S(E)$ we fix a place $w$ of $E$ above $v$, write $G_w$ for its decomposition subgroup in $G$ and (in the last isomorphism) use the $\mathcal{O}[G]$-module isomorphism

$$\mathcal{A}_F^n \otimes_{\mathcal{O}} H^2(E_v, T) \cong \mathcal{A}_F^n \otimes_{\mathcal{O}} \big(\mathcal{O}[G] \otimes_{\mathcal{O}[G]} H^2(E_w, T)\big) \cong \mathcal{O}[G] \otimes_{\mathcal{O}[G_w]} \big(\mathcal{A}_F^n \otimes_{\mathcal{O}} H^2(E_w, T)\big).$$

The above displayed isomorphism implies that the first term in the exact sequence in claim (i) is annihilated by the natural number $m$ specified in claim (ii). Thus, if $a$ is any element of $\mathcal{A}_F$ as in claim (ii), then the exact sequence in claim (i) implies that $a' := ma$ annihilates $H_0\big(G, \mathcal{A}_F^n \otimes_{\mathcal{O}} Ш\big) \cong \mathcal{A}_F^n \otimes_{\mathcal{O}[G]} Ш$. In view of the natural isomorphisms

$$H_0\big(G, \mathcal{A}_F^n \otimes_{\mathcal{O}} Ш\big)^\vee \cong H^0\big(G, (\mathcal{A}_F^n \otimes_{\mathcal{O}} Ш)^\vee\big) \cong H^0\big(G, (\mathcal{A}_F^n)^* \otimes_{\mathcal{O}} Ш^\vee\big)$$

one therefore has $a' \cdot H^0\big(G, (\mathcal{A}_F^n)^* \otimes_{\mathcal{O}} Ш^\vee\big) = (0)$.

With respect to the standard basis $\{b_i^*\}_{1\le i\le n}$, the action of $G$ on $(\mathcal{A}_F^n)^*$ is via the contragredient $\check{\varrho} : G \to \mathrm{GL}_n(\mathcal{A})$ of $\varrho$ (so $\check{\varrho}(g)$ is the transpose of $\varrho(g^{-1})$ for $g \in G$). Then, for each $x \in Ш^\vee$ the element $T_i(x) := \sum_{g\in G} g(b_i^* \otimes x)$ belongs to $H^0\big(G, (\mathcal{A}_F^n)^* \otimes_{\mathcal{O}} Ш^\vee\big)$ and so, in $(\mathcal{A}_F^n)^* \otimes_{\mathcal{O}} Ш^\vee$, one has

$$\begin{aligned} 0 = a'(T_i(x)) = a'\sum\nolimits_{g\in G}\big(b_i^* g^{-1} \otimes g(x)\big) &= a'\sum\nolimits_{g\in G}\big(\big(\sum\nolimits_{1\le j\le n}\check{\varrho}(g^{-1})_{ij} \cdot b_j^*\big) \otimes g(x)\big) \\ &= \sum\nolimits_{1\le j\le n}\big(\sum\nolimits_{g\in G}\big((a' \cdot \check{\varrho}(g^{-1})_{ij}) \cdot b_j^* \otimes g(x)\big)\big). \end{aligned}$$

For each $i$ and $j$ with $1 \le i, j \le n$, one therefore has

$$0 = \sum\nolimits_{g\in G}\big(a' \cdot \check{\varrho}(g^{-1})_{ij} \otimes g(x)\big) \in \mathcal{A}_F \otimes_{\mathcal{O}} Ш^\vee.$$

Upon taking images under the composite map

$$\mathcal{A}_F \otimes_{\mathcal{O}} Ш^\vee \xrightarrow{\epsilon\otimes\mathrm{id}} \mathcal{O}[G] \otimes_{\mathcal{O}} Ш^\vee \xrightarrow{g\otimes x\mapsto g(x)} Ш^\vee$$

one deduces $\sum_{g\in G}\epsilon(a' \cdot \check{\varrho}(g^{-1})_{ij})g(x)$ vanishes. Thus, since $x$ is an arbitrary element of $Ш^\vee$, each element $c(a')_{ij} := \sum_{g\in G}\epsilon(a' \cdot \check{\varrho}(g^{-1})_{ij})g$ of $\mathcal{O}[G]$ annihilates $Ш^\vee$. Hence, the sum

$$\begin{aligned} \sum\nolimits_{1\le i\le n} c(a')_{ii} = \sum\nolimits_{g\in G}\epsilon\big(a'\sum\nolimits_{1\le i\le n}\check{\varrho}(g^{-1})_{ii}\big)g &= \sum\nolimits_{g\in G}\epsilon\big(a' \cdot \chi_{\check{\varrho},F}(g^{-1})\big)g \\ &= \epsilon_G\big(\sum\nolimits_{g\in G}(a' \cdot \chi_{\varrho}(g) \otimes g)\big) = \epsilon_G(a' \cdot \mathrm{pr}_{\varrho,F}) \end{aligned}$$

also annihilates $Ш^\vee$, as claimed. □

## 4. Cohomological constructions of non-commutative Euler systems

In this section we present a general Galois-cohomological construction of non-commutative Euler systems and then use it to prove Theorems 2.19 and 2.22. In particular, throughout the section we assume the conditions stated in Theorem 2.19.

4.1. **Vertical reduced determinantal systems.** For each bounded below complex of $\mathcal{O}$-modules $C$ we set $C^* := \mathsf{R}\,\mathrm{Hom}_{\mathcal{O}}(C, \mathcal{O})$.

4.1.1. For each field $F$ in $\Omega(\mathcal{K}/K)$ and a finite set $\Sigma$ of places of $K$ with $S(F) \subset \Sigma$, we write $\mathsf{R}\Gamma_c(\mathcal{O}_{F,\Sigma}, T)$ for the compactly supported étale cohomology of $T$ on $\mathrm{Spec}(\mathcal{O}_{F,\Sigma})$.

We then define an object of $\mathsf{D}(\mathcal{A}[\mathcal{G}_F])$ by setting

$$C_{F,\Sigma}(T) := \mathsf{R}\Gamma_c(\mathcal{O}_{F,\Sigma}, T^*(1))^*[-2],$$

regarded as endowed with the natural action of $\mathcal{A}$ and the contragredient action of $\mathcal{G}_F$.

In the following result we use the full triangulated subcategory $\mathsf{D}^{\mathrm{lf},0}(\mathcal{A}[\mathcal{G}_F])$ of $\mathsf{D}(\mathcal{A}[\mathcal{G}_F])$ comprising complexes that are isomorphic to a bounded complex of finitely generated locally-free $\mathcal{A}[\mathcal{G}_F]$-modules whose Euler characteristic in $K_0^{\mathrm{lf}}(\mathcal{A}[\mathcal{G}_F])$ belongs to $\mathrm{SK}_0^{\mathrm{lf}}(\mathcal{A})$ (cf. [13, §5.1.4]).

**Lemma 4.1.** *For each $E$ in $\Omega(\mathcal{K}/K)$ and each finite set of places $\Sigma$ of $K$ that contains $S(E)$ the complex $C_{E,\Sigma}(T)$ has all of the following properties.*

(i) *$C_{E,\Sigma}(T)$ belongs to $\mathsf{D}^{\mathrm{lf},0}(\mathcal{A}[\mathcal{G}_E])$.*

(ii) *$C_{E,\Sigma}(T)$ is acyclic outside degrees zero and one and there is a canonical identification*

$$H^0(C_{E,\Sigma}(T)) = H^1(\mathcal{O}_{E,\Sigma}, T)$$

*and short exact sequence of $\mathcal{A}[\mathcal{G}_E]$-modules*

$$0 \to H^2(\mathcal{O}_{E,\Sigma}, T) \to H^1(C_{E,\Sigma}(T)) \to Y_E(T^*(1))^* \to 0.$$

(iii) *Given a finite set $\Sigma'$ of places of $K$ that contains $\Sigma$ there exists a canonical exact triangle in $\mathsf{D}^{\mathrm{perf}}(\mathcal{A}[\mathcal{G}_E])$ of the form*

$$\bigoplus\nolimits_{v\in\Sigma'\setminus\Sigma} \mathsf{R}\Gamma(K_v^{\mathrm{ur}}/K_v, T^*(1)_E)^*[-2] \to C_{E,\Sigma}(T) \to C_{E,\Sigma'}(T) \to .$$

(iv) *For all fields $E$ and $E'$ in $\Omega(\mathcal{K}/K)$ with $E \subseteq E'$ there exists a natural isomorphism $\mathcal{A}[\mathcal{G}_E] \otimes^{\mathsf{L}}_{\mathcal{A}[\mathcal{G}_{E'}]} C_{E',S(E')}(T) \cong C_{E,S(E')}(T)$ in $\mathsf{D}^{\mathrm{perf}}(\mathcal{A}[\mathcal{G}_E])$.*

*Proof.* To prove claim (i), it is enough (by [13, Rem. 5.3]) to show $C_{E,\Sigma}(T)$ belongs to $\mathsf{D}^{\mathrm{perf}}(\mathcal{A}[\mathcal{G}_E])$ and that its Euler characteristic $\chi^{\mathrm{proj}}_{\mathcal{A}[\mathcal{G}_E]}(C_{E,\Sigma}(T))$ in $K_0(\mathcal{A}[\mathcal{G}_E])$ vanishes.

To prove this we note Shapiro's Lemma identifies $\mathsf{R}\Gamma_c(\mathcal{O}_{E,\Sigma}, T^*(1))$ with the complex

$$C(E) := \mathsf{R}\Gamma_c(\mathcal{O}_{K,\Sigma}, T^*(1)_E).$$

In addition, since $T^*(1)_E$ is a projective $\mathcal{A}[\mathcal{G}_E]$-module (as a consequence of Hypothesis 2.3), the result of Flach [24, Th. 5.1] implies $C(E)$ belongs to $\mathsf{D}^{\mathrm{perf}}(\mathcal{A}[\mathcal{G}_E])$. This in turn implies that $C_{E,\Sigma}(T)$ belongs to $\mathsf{D}^{\mathrm{perf}}(\mathcal{A}[\mathcal{G}_E])$, since $\mathcal{A}[\mathcal{G}_E]$ is Gorenstein and so $\mathsf{D}^{\mathrm{perf}}(\mathcal{A}[\mathcal{G}_E])$ is preserved by the exact functor $C \mapsto C^*[-2]$.

For the same reason, to show that $\chi^{\mathrm{proj}}_{\mathcal{A}[\mathcal{G}_E]}(C_{E,\Sigma}(T))$ vanishes, it is enough to prove that $\chi^{\mathrm{proj}}_{\mathcal{A}[\mathcal{G}_E]}(C(E))$ vanishes. To do this, we set $\mathfrak{a} := \mathcal{A}/(p)$. Then, since $p$ is contained in the Jacobson radical of $\mathcal{A}[\mathcal{G}_E]$ the natural reduction map $K_0(\mathcal{A}[\mathcal{G}_E]) \to K_0(\mathfrak{a}[\mathcal{G}_E])$ is injective (by [2, Chap. IX, Prop. 1.3]) and so it is enough to note [24, Th. 5.1] also implies that the Euler characteristic in $K_0(\mathfrak{a}[\mathcal{G}_E])$ of the complex

$$\mathbb{Z}/(p) \otimes^{\mathsf{L}}_{\mathbb{Z}} C(E) \cong \mathsf{R}\Gamma_c(\mathcal{O}_{K,\Sigma}, (T^*(1)_E)/(p))$$

vanishes. This proves claim (i).

The fact that $C_{E,\Sigma}(T)$ is acyclic outside degrees zero and one is well-known and the existence of a canonical isomorphism and short exact sequence as in claim (ii) follows directly from the Artin-Verdier Duality theorem (and is also well-known).

The exact triangle in claim (iii) is obtained by applying the exact functor $X \mapsto X^*[-2]$ to the canonical exact triangle in $\mathsf{D}^{\mathrm{perf}}(\mathcal{A}[\mathcal{G}_E])$

$$\mathsf{R}\Gamma_c(\mathcal{O}_{E,\Sigma'}, T^*(1)) \to \mathsf{R}\Gamma_c(\mathcal{O}_{E,\Sigma}, T^*(1)) \to \bigoplus\nolimits_{v\in\Sigma'\setminus\Sigma} \mathsf{R}\Gamma(K_v^{\mathrm{ur}}/K_v, T^*(1)_E) \to .$$

The isomorphisms in claim (iv) result from combining the canonical isomorphisms

$$\mathcal{A}[\mathcal{G}_E] \otimes^{\mathsf{L}}_{\mathcal{A}[\mathcal{G}_{E'}]} C(E')^*[-2] \cong \mathsf{R}\,\mathrm{Hom}_{\mathcal{A}[\mathcal{G}_{E'}]}(\mathcal{A}[\mathcal{G}_E], C(E'))^*[-2]$$

and $\mathsf{R}\,\mathrm{Hom}_{\mathcal{A}[\mathcal{G}_{E'}]}(\mathcal{A}[\mathcal{G}_E], C(E')) \cong C(E)$. □

**Remark 4.2.** If $T$ arises as the $p$-adic Tate module of a critical motive, then the complexes $C_{F,\Sigma}(T)$ can also usefully be interpreted in terms of the formalism of 'Selmer complexes' developed by Nekovář in [36].

4.1.2. For a commutative noetherian ring $\Lambda$ we write $\mathcal{P}(\Lambda)$ for the category of graded invertible $\Lambda$-modules (as reviewed in [13, §5.1.1]).

In the sequel, for each $\chi$ in $\mathrm{Ir}_p(K)$ we use the approach of §2.1.1 to fix an ordered $\mathbb{Q}_p^c$-basis of a $\mathbb{Q}_p^c[\mathcal{G}_{K(\chi)}]$-module of character $\chi$. Then, for each $F$ in $\Omega(\mathcal{K}/K)$, this data gives rise to a collection $\varpi_F$ of ordered $\mathbb{Q}_p^c$-bases of a complete set of representatives of the isomorphism classes of simple $(\mathbb{Q}_p^c \otimes_{\mathcal{Q}} A)[\mathcal{G}_F]$-modules. We shall use the associated $\mathcal{P}(\xi(\mathcal{A}[\mathcal{G}_F]))$-valued reduced determinant functor

$$\mathrm{d}_{\mathcal{A}[\mathcal{G}_F]}(-) = \mathrm{d}_{\mathcal{A}[\mathcal{G}_F],\varpi_F}(-)$$

that is constructed in [13, Th. 5.4].

**Remark 4.3.** For $F$ in $\Omega(\mathcal{K}/K)$, the approach of Deligne in [21, §4] constructs a 'universal determinant functor' for the (exact) category of finitely-generated locally-free $\mathcal{A}[\mathcal{G}_F]$-modules, with values in an associated commutative Picard category $\mathcal{V}^{\mathrm{lf}}(\mathcal{A}[\mathcal{G}_F])$ of 'virtual objects'. The functor $\mathrm{d}_{\mathcal{A}[\mathcal{G}_F]}(-)$ therefore induces a strongly monoidal functor from $\mathcal{V}^{\mathrm{lf}}(\mathcal{A}[\mathcal{G}_F])$ to $\mathcal{P}(\xi(\mathcal{A}[\mathcal{G}_F]))$. We recall that, in most cases, the latter functor is not an equivalence of commutative Picard categories (for details see [13, Rems. 5.5 and 5.8]).

For each pair of fields $F$ and $F'$ in $\Omega(\mathcal{K}/K)$ with $F \subseteq F'$ we write

$$\nu_{F'/F} : \mathrm{d}_{\mathcal{A}[\mathcal{G}_{F'}]}(C_{F',S(F')}(T)) \to \mathrm{d}_{\mathcal{A}[\mathcal{G}_F]}(C_{F,S(F)}(T))$$

for the composite surjective homomorphism of (graded) $\xi(\mathcal{A}[\mathcal{G}_{F'}])$-modules

$$\begin{aligned} & \mathrm{d}_{\mathcal{A}[\mathcal{G}_{F'}]}(C_{F',S(F')}(T)) \\ \to\ & \xi(\mathcal{A}[\mathcal{G}_F]) \otimes_{\xi(\mathcal{A}[\mathcal{G}_{F'}])} \mathrm{d}_{\mathcal{A}[\mathcal{G}_{F'}]}(C_{F',S(F')}(T)) \\ \cong\ & \mathrm{d}_{\mathcal{A}[\mathcal{G}_F]}(C_{F,S(F')}(T)) \\ \cong\ & \mathrm{d}_{\mathcal{A}[\mathcal{G}_F]}(C_{F,S(F)}(T)) \otimes \bigotimes_{v\in S(F')\setminus S(F)} \mathrm{d}_{\mathcal{A}[\mathcal{G}_F]}(\mathsf{R}\Gamma(K_v^{\mathrm{ur}}/K_v, T^*(1)_F)^*[-2])^{-1} \\ \cong\ & \mathrm{d}_{\mathcal{A}[\mathcal{G}_F]}(C_{F,S(F)}(T)). \end{aligned}$$

Here the first arrow denotes the natural projection map, the first isomorphism is induced by the quasi-isomorphism in Lemma 4.1(iii) and the general result of [13, Th. 5.4(ii)] and the second isomorphism is induced by the exact triangle in Lemma 4.1(ii) and the result of [13, Th. 5.4(i)]. Finally, the third isomorphism is induced by the fact that for each $v$ in $S(F') \setminus S(F)$ the complex $\mathsf{R}\Gamma(K_v^{\mathrm{ur}}/K_v, T^*(1)_F)^*[-2]$ is represented by

$$(T^*(1)_F)^* \xrightarrow{1-\sigma_v} (T^*(1)_F)^*, \tag{4.1.1}$$

where the first term is in degree one, and so there is a canonical composite isomorphism

$$\begin{aligned}(4.1.2)\quad & \mathrm{d}_{\mathcal{A}[\mathcal{G}_F]}(\mathsf{R}\Gamma(K_v^{\mathrm{ur}}/K_v, T^*(1)_F)^*[-2])^{-1} \\ &\cong \mathrm{d}^{\diamond}_{\mathcal{A}[\mathcal{G}_F]}((T^*(1)_F)^*) \otimes \mathrm{d}^{\diamond}_{\mathcal{A}[\mathcal{G}_F]}((T^*(1)_F)^*)^{-1} \xrightarrow{\sim} (\xi(\mathcal{A}[\mathcal{G}_F]), 0).\end{aligned}$$

Here the first isomorphism follows by applying [13, (5.3.1)] to the resolution (4.1.1) and the second is the natural 'evaluation map'.

**Definition 4.4.** For any Galois extension $\mathcal{K}$ of $K$ in $K^c$ the module of *vertical reduced determinantal systems* for $(T, \mathcal{K})$ is the inverse limit

$$\mathrm{VS}(T, \mathcal{K}) := \varprojlim_{F \in \Omega(\mathcal{K}/K)} \mathrm{d}_{\mathcal{A}[\mathcal{G}_F]}(C_{F,S(F)}(T)),$$

where the transition morphism for $F$ and $F'$ in $\Omega(\mathcal{K}/K)$ with $F \subset F'$ is $\nu_{F'/F}$. We refer to an element of $\mathrm{VS}(T, \mathcal{K})$ as a 'vertical reduced determinantal system' for the pair $(T, \mathcal{K})$.

**Remark 4.5.** If $\mathcal{K}/K$ is abelian, the above definition recovers the module $\mathrm{VS}(T, \mathcal{K})$ discussed in [12, §2.4]. We also recall that the terminology of 'vertical determinantal systems' is introduced in [12] in order to contrast these systems with the 'horizontal determinantal systems' that play a key role in loc. cit. (but for which we currently know of no non-commutative analogue).

It is clear that $\mathrm{VS}(T, \mathcal{K})$ has a natural action of the algebra $\xi(\mathcal{A}[[\mathrm{Gal}(\mathcal{K}/K)]])$ and the following result describes this structure explicitly.

**Proposition 4.6.** *The $\xi(\mathcal{A}[[\mathrm{Gal}(\mathcal{K}/K)]])$-module $\mathrm{VS}(T, \mathcal{K})$ is free of rank one.*

*Proof.* For each natural number $n$ we write $K_{(n)}$ for the composite of all finite extensions of $K$ inside $\mathcal{K}$ with the property that the absolute value of the discriminant of $K/\mathbb{Q}$ is at most $n$.

Then $K_{(n)}/K$ is a Galois extension and has finite degree as a consequence of the Hermite-Minkowski Theorem (cf. [37, §III.2]). In addition, one has $K_{(n)} \subset K_{(n+1)}$ for all $n$ and the normal subgroups $\{\mathrm{Gal}(\mathcal{K}/K_{(n)})\}_{n\geq 1}$ form a base of neighbourhoods of the identity in $\mathrm{Gal}(\mathcal{K}/K)$. Thus, if for each $n$ we set $\mathcal{G}_n := \mathcal{G}_{K_{(n)}}$ and $\Xi_n := \mathrm{d}_{\mathcal{A}[\mathcal{G}_n]}(C_{K_{(n)},S(K_{(n)})}(T))$ and write $\tau_n$ for the (surjective) transition morphism $\Xi_{n+1} \to \Xi_n$ used in the definition of $\mathrm{VS}(T, \mathcal{K})$, then there is a canonical identification

$$\mathrm{VS}(T, \mathcal{K}) = \varprojlim_{n\geq 1} \Xi_n, \tag{4.1.3}$$

where the limit is taken with respect to the morphisms $\tau_n$.

For every $n$ the $\xi(\mathcal{A}[\mathcal{G}_n])$-module $\Xi_n$ is, by construction, free of rank one. We may therefore assume that, for some fixed $n$, we have made a choice for each natural number $m$ with $m \leq n$ of an $\xi(\mathcal{A}[\mathcal{G}_m])$-basis $x_m$ of $\Xi_m$ so that $\tau_m(x_{m+1}) = x_m$ for all $m < n$.

If now $x'_{n+1}$ is any choice of $\xi(\mathcal{A}[\mathcal{G}_{n+1}])$-basis of $\Xi_{n+1}$, then $\tau_n(x'_{n+1})$ is a $\xi(\mathcal{A}[\mathcal{G}_n])$-basis of $\Xi_n$ and so there exists a unit $u_n$ of $\xi(\mathcal{A}[\mathcal{G}_n])$ such that $x_n = u_n \cdot \tau_n(x'_{n+1})$.

But, since $\xi(\mathcal{A}[\mathcal{G}_{n+1}])$ is semi-local and the projection map $\xi(\mathcal{A}[\mathcal{G}_{n+1}]) \to \xi(\mathcal{A}[\mathcal{G}_n])$ is surjective (by [13, Lem. 3.2(v)]), Bass' Theorem (cf. [31, Chap. 7, (20.9)]) implies that the homomorphism $\xi(\mathcal{A}[\mathcal{G}_{n+1}])^\times \to \xi(\mathcal{A}[\mathcal{G}_n])^\times$ induced by the projection map is also surjective and so we may fix a pre-image $u_{n+1}$ of $u_n$ under this map.

It is then easily checked that the element $x_{n+1} := u_{n+1} \cdot x'_{n+1}$ is a $\xi(\mathcal{A}[\mathcal{G}_{n+1}])$-basis of $\Xi_{n+1}$ with the property that $(\tau_m \circ \tau_{m+1} \circ \cdots \circ \tau_n)(x_{n+1}) = x_m$ for all $m < n+1$.

Continuing in this way we inductively define an element $(x_n)_{n\geq 1}$ that the isomorphism (4.1.3) implies is a $\xi(\mathcal{A}[[\mathrm{Gal}(\mathcal{K}/K)]])$-basis of $\mathrm{VS}(T,\mathcal{K})$, as required. □

4.2. **Reduced determinants and non-commutative Euler systems.** In this section we describe the crucial link between reduced determinantal systems and non-commutative Euler systems.

4.2.1. At the outset we note that condition (c) of Theorem 2.19 combines with Hypothesis 2.1 to imply that the $\mathcal{A}$-module $Y_K(T^*(1))^*$ is free and we fix an (ordered) basis $\{a_i\}_{1\leq i\leq r}$.

In addition, for each $E$ in $\Omega(\mathcal{K}/K)$ there is a decomposition of $\mathcal{A}[\mathcal{G}_E]$-modules

$$Y_E(T^*(1)) = \bigoplus\nolimits_{v\in S_\infty(K)} \left(\bigoplus\nolimits_{w|v} H^0(E_w, T^*(1))\right)$$

where $w$ runs over all places of $E$ above $v$. This decomposition implies, in particular, that if we assume condition (d) of Theorem 2.19 and then fix a set of representatives of the $G_K$-orbits of embeddings $\mathcal{K} \to \mathbb{Q}^c$, we obtain (by restriction of the embeddings) a compatible family of isomorphisms of $\mathcal{A}[\mathcal{G}_E]$-modules

$$Y_E(T^*(1))^* \cong \mathcal{A}[\mathcal{G}_E] \otimes_{\mathcal{A}} Y_K(T^*(1))^* \cong \mathcal{A}[\mathcal{G}_E]^r, \tag{4.2.1}$$

where the second map is induced by the chosen $\mathcal{A}$-basis $\{a_i\}_{1\leq i\leq r}$ of $Y_K(T^*(1))^*$.

**Theorem 4.7.** *If the conditions (a), (b), (c) and (d) of Theorem 2.19 are satisfied, then for each fixed set of isomorphisms (4.2.1) as above there exists a canonical homomorphism of $\xi(\mathcal{A}[[\mathrm{Gal}(\mathcal{K}/K)]])$-modules $\Theta_{T,\mathcal{K}} : \mathrm{VS}(T,\mathcal{K}) \to \mathrm{ES}_r(T,\mathcal{K})$.*

*This homomorphism is non-zero if and only if there exists a field $F$ in $\Omega(\mathcal{K}/K)$ and a non-zero primitive idempotent $e$ of $\zeta(A[\mathcal{G}_F])$ for which the space $e\cdot H^2(\mathcal{O}_{F,S(F)}, V)$ vanishes.*

**Remark 4.8.** A conjecture of Jannsen [28, Conj. 1] implies that, for any given $F$ in $\Omega(\mathcal{K}/K)$, the space $H^2(\mathcal{O}_{F,S(F)}, V)$ should vanish for all but a few exceptional representations $V$.

The proof of Theorem 4.7 will occupy the rest of this section. The basic strategy will be to define for each $L$ in $\Omega(\mathcal{K}/K)$ a canonical homomorphism of $\xi(\mathcal{A}[[\mathrm{Gal}(\mathcal{K}/K)]])$-modules

$$\Theta_L : \mathrm{VS}(T,\mathcal{K}) \to \mathcal{Q} \otimes_{\mathcal{O}} {\bigcap}^r_{\mathcal{A}[\mathcal{G}_L]} H^1(\mathcal{O}_{L,S(L)}, T)$$

and then to prove that the image of the diagonal homomorphism

$$\Theta_{T,\mathcal{K}} : \mathrm{VS}(T,\mathcal{K}) \xrightarrow{(\Theta_L)_L} \prod\nolimits_{L\in\Omega(\mathcal{K}/K)} \big(\mathcal{Q} \otimes_{\mathcal{O}} {\bigcap}^r_{\mathcal{A}[\mathcal{G}_L]} H^1(\mathcal{O}_{L,S(L)}, T)\big)$$

belongs to $\mathrm{ES}_r(T,\mathcal{K})$ and to determine when this homomorphism is zero.

4.2.2. In this section we use the idempotent $e_L$ of $\zeta(A[\mathcal{G}_L])$ defined in (2.2.5). We also set

$$C_{L,S(L)}(V) := \mathcal{Q} \otimes_{\mathcal{O}} C_{L,S(L)}(T)$$

**Lemma 4.9.** *For each $L$ in $\Omega(\mathcal{K}/K)$ set $\mathrm{rr}_L := \mathrm{rr}_{A[\mathcal{G}_L]e_L}((A[\mathcal{G}_L]e_L)^r)$. Then the following claims are valid.*

(i) *The isomorphism (4.2.1) induces an isomorphism in $\mathcal{P}(\zeta(A[\mathcal{G}_L])e_L)$*

$$\mathrm{d}^{\diamond}_{A[\mathcal{G}_L]e_L}(e_L \cdot H^1(C_{L,S(L)}(V))) \cong (\zeta(A[\mathcal{G}_L])e_L, \mathrm{rr}_L).$$

(ii) *There is an inclusion*

$$\mathrm{d}^{\diamond}_{A[\mathcal{G}_L]e_L}(e_L \cdot H^0(C_{L,S(L)}(V))) \subseteq e_L(\mathcal{Q} \otimes_{\mathcal{O}} {\bigcap}^r_{\mathcal{A}[\mathcal{G}_L]} H^1(\mathcal{O}_{L,S(L)}, T), \mathrm{rr}_L).$$

*Proof.* Set $\mathcal{A}_L := \mathcal{A}[\mathcal{G}_L]$, $A_L := A[\mathcal{G}_L]$ and $\mathrm{rr}'_L := \mathrm{rr}_{A_L}(A_L^r)$.

To prove claim (i) we note that the definition of $e_L$ combines with the exact sequence in Lemma 4.1(ii) to give an identification of spaces $e_L \cdot H^1(C_{L,S(L)}(V)) = e_L(\mathcal{Q} \otimes_{\mathcal{O}} Y_L(T^*(1))^*)$ and hence also an identification in $\mathcal{P}(\zeta(A_L)e_L)$

$$\begin{aligned}\mathrm{d}^{\diamond}_{A_Le_L}(e_L \cdot H^1(C_{L,S(L)}(V))) &= \mathrm{d}^{\diamond}_{A_Le_L}(e_L(\mathcal{Q} \otimes_{\mathcal{O}} Y_L(T^*(1))^*)) \\ &= e_L(\mathcal{Q} \otimes_{\mathcal{O}} \mathrm{d}^{\diamond}_{\mathcal{A}_L}(Y_L(T^*(1))^*)).\end{aligned}$$

Given this, the isomorphism in claim (i) is induced by the isomorphism in $\mathcal{P}(\xi(\mathcal{A}_L))$

$$(\xi(\mathcal{A}_L), \mathrm{rr}'_L) \cong \big({\bigcap}^r_{\mathcal{A}_L} Y_L(T^*(1))^*, \mathrm{rr}'_L\big) = \mathrm{d}^{\diamond}_{\mathcal{A}_L}(Y_L(T^*(1))^*)$$

obtained by applying the result of [13, Prop. 5.9(i)] to the module $M = Y_L(T^*(1))^*$ with $\{b_j\}_{1\le j\le r}$ equal to the basis that (4.2.1) sends to the standard basis of $\mathcal{A}_L^r$.

To prove claim (ii) we note that Lemma 4.1(i) and (ii) combine to imply the existence of an $\mathcal{A}_L$-submodule $X$ of $H^1(\mathcal{O}_{L,S(L)}, T)$ that is free of rank $r$ and such that $e_L \cdot (\mathcal{Q} \otimes_{\mathcal{O}} X) = e_L \cdot H^0(C_{L,S(L)}(V))$. Then, just as above, one has

$$\begin{aligned}\mathrm{d}^{\diamond}_{A_Le_L}(e_L \cdot H^0(C_{L,S(L)}(V))) &= e_L \cdot (\mathcal{Q} \otimes_{\mathcal{O}} \mathrm{d}^{\diamond}_{\mathcal{A}_L}(X)) \\ &= \big(e_L\big(\mathcal{Q} \otimes_{\mathcal{O}} {\bigcap}^r_{\mathcal{A}_L} X\big), \mathrm{rr}_L\big) \\ &\subseteq \big(e_L\big(\mathcal{Q} \otimes_{\mathcal{O}} {\bigcap}^r_{\mathcal{A}_L} H^1(\mathcal{O}_{L,S(L)}, T)\big), \mathrm{rr}_L\big),\end{aligned}$$

where the final inclusion follows from the general result of [13, Th. 4.19(iv)]. This proves claim (ii). □

We define $\Theta'_L$ to be the composite homomorphism of $\zeta(A[\mathcal{G}_L])$-modules

$$\begin{aligned}&\mathrm{d}_{A[\mathcal{G}_L]}(C_{L,S(L)}(V)) \\ &\cong \mathrm{d}^{\diamond}_{A[\mathcal{G}_L]}(H^0(C_{L,S(L)}(V))) \otimes \mathrm{d}^{\diamond}_{A[\mathcal{G}_L]}(H^1(C_{L,S(L)}(V)))^{-1} \\ &\to \mathrm{d}^{\diamond}_{A[\mathcal{G}_L]e_L}(e_L \cdot H^0(C_{L,S(L)}(V))) \otimes \mathrm{d}^{\diamond}_{A[\mathcal{G}_L]e_L}(e_L \cdot H^1(C_{L,S(L)}(V)))^{-1} \\ &\cong \mathrm{d}^{\diamond}_{A[\mathcal{G}_L]e_L}(e_L \cdot H^0(C_{L,S(L)}(V))) \\ &\to e_L\big(\mathcal{Q} \otimes_{\mathcal{O}} {\bigcap}^r_{\mathcal{A}[\mathcal{G}_L]} H^1(\mathcal{O}_{L,S(L)}, T)\big),\end{aligned} \tag{4.2.2}$$

where the first map is the 'passage to cohomology' map from [13, Prop. 5.17(i)], the second is induced by multiplication by $e_L$ and the final two by the results in Lemma 4.9.

We can now finally define $\Theta_L$ to be the composite homomorphism

$$\begin{aligned}\mathrm{VS}(T,\mathcal{K}) \to \mathrm{d}_{\mathcal{A}[\mathcal{G}_L]}(C_{L,S(L)}(T)) \subset \mathrm{d}_{\mathcal{A}[\mathcal{G}_L]}(C_{L,S(L)}(V))&\\ \xrightarrow{\Theta'_L} e_L\left(\mathcal{Q}\otimes_{\mathcal{O}}\bigcap\nolimits^r_{\mathcal{A}[\mathcal{G}_L]} H^1(\mathcal{O}_{L,S(L)},T)\right),&\end{aligned}$$

where the first arrow is the canonical projection.

Then we need to prove that this definition implies that for every $\eta$ in $\mathrm{VS}(T,\mathcal{K})$ and every pair of fields $F$ and $F'$ in $\Omega(\mathcal{K}/K)$ with $F\subset F'$ one has both

$$\Theta_{F'}(\eta) \in \bigcap\nolimits^r_{\mathcal{A}[\mathcal{G}_{F'}]} H^1(\mathcal{O}_{F',S(F')},T) \tag{4.2.3}$$

and

$$\mathrm{Cor}^r_{F'/F}(\Theta_{F'}(\eta)) = \left(\prod\nolimits_{v\in\Sigma} P_v\right)\cdot \Theta_F(\eta), \tag{4.2.4}$$

with $\Sigma := S(F')\setminus S(F)$ and $P_v := \mathrm{Nrd}_{\mathcal{A}[\mathcal{G}_F]}(1-\sigma_v \mid V^*(1)_F)^{\#}$ for each $v\in\Sigma$.

4.2.3. The next result plays a key role in the proof of these facts. In order to state it, for any ring $\Lambda$ and natural numbers $d$ and $d'$, we identify each matrix $M$ in $\mathrm{M}_{d',d}(\Lambda)$ with the homomorphism of $\mathcal{A}$-modules $\theta_M : \Lambda^{d'} \to \Lambda^d$ that sends each (row) vector $x$ to $x\cdot M$.

**Lemma 4.10.** *For each $L$ in $\Omega(\mathcal{K}/K)$ there exists an exact sequence of $\mathcal{A}[\mathcal{G}_L]$-modules*

$$0 \to H^1(\mathcal{O}_{L,S(L)},T) \xrightarrow{\iota_L} P_L \xrightarrow{\theta_L} P_L \xrightarrow{\pi_L} H^1(C_{L,S(L)}(T)) \to 0 \tag{4.2.5}$$

*that satisfies all of the following properties.*

(i) *$P_L$ is finitely generated and free of rank $d_L > r$.*

(ii) *Consider the composite surjective homomorphism of $\mathcal{A}[\mathcal{G}_L]$-modules*

$$P_L \xrightarrow{\pi_L} H^1(C_{L,S(L)}(T)) \to Y_L(T^*(1))^* \cong \mathcal{A}[\mathcal{G}_L]^r,$$

*where the second map comes from the exact sequence in Lemma 4.1(ii) and the isomorphism is induced by (4.2.1) and Hypothesis 2.1. Then there exists an ordered basis $\{b_{i,L}\}_{1\le i\le d_L}$ of $P_L$ such that the above homomorphism sends $b_{i,L}$ to the $i$-th element of the standard basis of $\mathcal{A}[\mathcal{G}_L]^r$ if $1\le i\le r$ and to zero otherwise.*

(iii) *Write $P_L^\bullet$ for the complex $P_L \to P_L$, where the first term is placed in degree zero, the differential is $\theta_L$ and $H^0(P_L^\bullet)$ and $H^1(P_L^\bullet)$ are identified with $H^0(C_{L,S(L)}(T)) = H^1(\mathcal{O}_{L,S(L)},T)$ and $H^1(C_{L,S(L)}(T))$ by using the maps $\iota_L$ and $\pi_L$. Then there exists an isomorphism in $\mathsf{D}^{\mathrm{perf}}(\mathcal{A}[\mathcal{G}_L])$ from $P_L^\bullet$ to $C_{L,S(L)}(T)$ that induces the identity map in all degrees of cohomology.*

(iv) *The matrix in $\mathrm{M}_{d_L}(\mathcal{A}[\mathcal{G}_L])$ that represents $\theta_L$ with respect to the basis $\{b_{i,L}\}_{1\le i\le d_L}$ is a block matrix $\left(0_{d_L,r} \mid M_L\right)$ where $M_L$ belongs to $\mathrm{M}_{d_L,d_L-r}(\mathcal{A}[\mathcal{G}_L])$ and is such that* $\ker(\theta_{M_L}) = H^1(\mathcal{O}_{L,S(L)},T)$.

*Proof.* Claims (i) and (ii) of Lemma 4.1 combine to imply that the $(-1)$-shift of $C_{L,S(L)}(T)$ is an 'admissible' complex in the sense of [9]. Given this fact, the existence of an exact sequence with all of the properties in (i), (ii) and (iii) follows from the general construction of [9, §3.1].

The property in claim (ii) implies that $\mathrm{im}(\theta_L)$ is contained in the submodule of $P_L$ generated by $\{b_{i,L}\}_{r<i\le d_L}$ and hence that $\theta_L$ is represented by a block matrix $M'_L$ of the form $\big(0_{d_L,r}\mid M_L\big)$ stated in claim (iv). For this representation it is then also clear that $H^1(\mathcal{O}_{L,S(L)},T)=\ker(\theta_L)$ is equal to $\ker(\theta_{M_L})$, as required. □

Since the image of the endomorphism $\theta=\theta_{F'}$ in Lemma 4.10 (with $L=F'$) is $\mathcal{O}$-free, and the algebra $\mathcal{A}$ is Gorenstein, the group $\mathrm{Ext}^1_{\mathcal{A}[\mathcal{G}_{F'}]}(\mathrm{im}(\theta),\mathcal{A}[\mathcal{G}_{F'}])$ vanishes (cf. Remark 2.2). The construction of [13, Prop. 4.21] can therefore be applied to the matrix $M=M_{F'}$ in Lemma 4.10(iv).

In view of the latter result, the containment (4.2.3) will follow if we can show the existence of an element $x_\eta$ of $\xi(\mathcal{A}[\mathcal{G}_{F'}])$ such that

$$\Theta_{F'}(\eta)=x_\eta\cdot\varepsilon_M, \tag{4.2.6}$$

where $\varepsilon_M$ is the canonical element constructed in [13, Prop. 4.21].

To prove this we note that claims (i) and (ii) of Lemma 4.1 combine to imply the existence of an (in general, non-canonical) isomorphism of $A[\mathcal{G}_{F'}]e_{F'}$-modules

$$e_{F'}\cdot H^1(\mathcal{O}_{F',S(F')},V)\cong e_{F'}\cdot(\mathcal{Q}\otimes_{\mathcal{O}}Y_{F'}(T^*(1))^*)\cong(A[\mathcal{G}_{F'}]e_{F'})^r \tag{4.2.7}$$

so that $e_{F'}\cdot H^1(\mathcal{O}_{F',S(F')},V)$ is a free $A[\mathcal{G}_{F'}]e_{F'}$-module of rank $r$. In addition, the elements $\Theta_{F'}(\eta)$ and $\varepsilon_M$ both belong to

$$e_{F'}\cdot{\bigwedge}^r_{A[\mathcal{G}_{F'}]}H^1(\mathcal{O}_{F',S(F')},V)={\bigwedge}^r_{A[\mathcal{G}_{F'}]e_{F'}}e_{F'}\cdot H^1(\mathcal{O}_{F',S(F')},V).$$

Upon combining the general result of [13, Lem. 4.12] with the argument of [13, Th. 4.19(ii)], one finds that the equality (4.2.6) is valid if $\Theta_{F'}(\eta)$ and $x_\eta\cdot\varepsilon_M$ have the same image under $\wedge_{i=1}^{i=r}\phi_i$ for every $\phi_i$ in $\mathrm{Hom}_{\mathcal{A}[\mathcal{G}_{F'}]}(P_{F'},\mathcal{A}[\mathcal{G}_{F'}])$, where we do not distinguish between $\phi_i$ and its restriction through $\iota_{F'}$ to $H^1(\mathcal{O}_{F',S(F')},T)$.

To check this we note that, setting $d:=d_{F'}$, the explicit definition of $\varepsilon_M$ implies (via [13, Lem. 4.10]) that

$$(\wedge_{i=1}^{i=r}\phi_i)(\varepsilon_M)=\mathrm{Nrd}_{\mathcal{A}[\mathcal{G}_{F'}]}\big((M'\mid M)\big) \tag{4.2.8}$$

where $M'=M'(\{\phi_i\}_{1\le i\le r})$ is the matrix in $\mathrm{M}_{d,r}(\mathcal{A}[\mathcal{G}_{F'}])$ with $ji$-entry equal to $\phi_i(b_{j,F'})$.

On the other hand, if we set $C_{F'}:=C_{F',S(F')}(T)$ and $\beta_{F'}:=((\wedge_{i=1}^{i=d}b_{i,F'})\otimes(\wedge_{i=1}^{i=d}b^*_{i,F'}),0)$, then Lemma 4.10(iii) allows us to identify $\mathrm{d}_{\mathcal{A}[\mathcal{G}_{F'}]}(C_{F'})$ with

$$\mathrm{d}_{\mathcal{A}[\mathcal{G}_{F'}]}(P^\bullet_{F'})=\mathrm{d}^\diamond_{\mathcal{A}[\mathcal{G}_{F'}]}(P_{F'})\otimes\mathrm{Hom}_{\xi(\mathcal{A}[\mathcal{G}_{F'}])}\big(\mathrm{d}^\diamond_{\mathcal{A}[\mathcal{G}_{F'}]}(P_{F'}),\xi(\mathcal{A}[\mathcal{G}_{F'}])\big)=\xi(\mathcal{A}[\mathcal{G}_{F'}])\cdot\beta_{F'}$$

(where the first equality follows from [13, (5.3.1)] and the second from [13, Prop. 5.9(i)]), and then the argument of [5, Lem. 7.3.1] implies that

$$(\wedge_{i=1}^{i=r}\phi_i)\big(\Theta'_{F'}(\beta_{F'})\big)=\mathrm{Nrd}_{\mathcal{A}[\mathcal{G}_{F'}]}\big((M'\mid M)\big).$$

This equality combines with (4.2.8) to imply that the equality (4.2.6), and hence also the containment (4.2.3), is valid since the image of any element $\eta$ of $\mathrm{VS}(T,\mathcal{K})$ in $\mathrm{d}_{\mathcal{A}[\mathcal{G}_{F'}]}(C_{F'})=\mathrm{d}_{\mathcal{A}[\mathcal{G}_{F'}]}(P^\bullet_{F'})$ is of the form $x_\eta\cdot\beta_{F'}$ for a unique element $x_\eta$ of $\xi(\mathcal{A}[\mathcal{G}_{F'}])$.

To prove (4.2.4) it is enough to prove an equality of maps

$$\mathrm{Cor}^r_{F'/F}\circ\Theta'_{F'}=\Big(\prod\nolimits_{v\in\Sigma}P_v\Big)\cdot(\Theta_F\circ\nu_{F'/F}). \tag{4.2.9}$$

To do this we set $\Delta := \mathrm{Gal}(F'/F)$ and write $T_\Delta$ for the (central) element $\sum_{\delta\in\Delta}\delta$ of $\mathcal{A}[\mathcal{G}_{F'}]$. We then identify $A[\mathcal{G}_F]$ with the subalgebra $T_\Delta \cdot A[\mathcal{G}_{F'}]$ of $A[\mathcal{G}_{F'}]$.

Then, since an idempotent $e$ of $\zeta(A[\mathcal{G}_F])$ annihilates $\mathrm{im}(\Theta'_{F'})$, respectively $\mathrm{im}(\Theta'_F)$, if and only if $e\cdot e_{F'} = 0$, respectively $e\cdot e_F = 0$, the result of Lemma 4.11 below implies it is enough to verify the above equality after multiplying by a primitive idempotent $e$ of $\zeta(A[\mathcal{G}_F])$ with the property that $e \cdot P_v \neq 0$ for all $v$ in $\Sigma$.

But, for any such $e$, the required equality is true since the result of Lemma 4.12 below combines with the explicit definitions of the maps $\Theta'_{F'}, \nu_{F'/F}$ and $\Theta'_F$ to imply the commutativity of the diagram

$$\begin{tikzcd}
e\cdot \mathrm{d}_{A[\mathcal{G}_{F'}]}(C_{F',S(F')}(V))) \arrow[r, "\Theta'_{F'}"] \arrow[d, "\nu_{F'/F}"'] & e\cdot {\bigwedge}^r_{A[\mathcal{G}_{F'}]} H^1(\mathcal{O}_{F',S(F')}, V) \arrow[d, "\mathrm{Cor}^r_{F'/F}"] \\
e\cdot \mathrm{d}_{A[\mathcal{G}_F]}(C_{F,S(F)}(V)) \arrow[r, "(\prod_{v\in\Sigma} P_v)\times\Theta'_F"'] & e\cdot {\bigwedge}^r_{A[\mathcal{G}_F]} H^1(\mathcal{O}_{F,S(F')}, V).
\end{tikzcd}$$

At this stage we have shown that the image of the diagonal map $\Theta_{T,\mathcal{K}} = (\Theta_L)_L$ is contained in $\mathrm{ES}_r(T,\mathcal{K})$ and so to complete the proof of Theorem 4.7 it is enough to prove its final claim. However, this claim is equivalent to asserting that $\Theta_{T,\mathcal{K}}$ is non-zero if and only if there exists a field $F'$ in $\Omega(\mathcal{K}/K)$ for which $e_{F'}$ is non-zero and this follows directly from the fact that the isomorphism (4.2.7) implies that the annihilator in $\zeta(A[\mathcal{G}_{F'}])$ of the image of $\Theta'_{F'}$ is equal to $\zeta(A[\mathcal{G}_{F'}])(1-e_{F'})$. This completes the proof of Theorem 4.7.

**Lemma 4.11.** *For each primitive idempotent $e$ of $\zeta(A[\mathcal{G}_F])$ one has $e\cdot e_{F'} \neq 0$ if and only if both $e\cdot P_v \neq 0$ for all $v$ in $\Sigma$ and also $e\cdot e_F \neq 0$.*

*Proof.* Claims (ii) and (iv) of Lemma 4.1 combine to induce an identification

$$H^2(\mathcal{O}_{F,S(F')}, V) \cong A[\mathcal{G}_F] \otimes_{A[\mathcal{G}_{F'}]} H^2(\mathcal{O}_{F',S(F')}, V) = T_\Delta \cdot H^2(\mathcal{O}_{F',S(F')}, V)$$

and so the definition of $e_{F'}$ implies that $e\cdot e_{F'} \neq 0$ if and only if $e\cdot H^2(\mathcal{O}_{F,S(F')}, V) = 0$.

To study this condition we set $W_F := \mathrm{Hom}_{\mathcal{Q}}(V^*(1)_F, \mathcal{Q})$ and note that the exact cohomology sequence of the triangle in Lemma 4.1(iii) gives rise to an exact sequence

$$\bigoplus_{v\in\Sigma} \ker\big(\phi_v \mid W_F\big) \xrightarrow{\theta} H^2(\mathcal{O}_{F,S(F)}, V) \to H^2(\mathcal{O}_{F,S(F')}, V) \to \bigoplus_{v\in\Sigma} \mathrm{cok}\big(\phi_v \mid W_F\big) \to 0,$$

where we set $\phi_v := 1-\sigma_v$.

This sequence implies $e\cdot H^2(\mathcal{O}_{F,S(F')}, V) = 0$ if and only if both $e\cdot\mathrm{cok}\big(\phi_v \mid W_F\big) = 0$ for all $v\in\Sigma$ and also $e\cdot\mathrm{cok}(\theta) = 0$. In addition, for each such $v$ the condition $e\cdot\mathrm{cok}\big(\phi_v \mid W_F\big) = 0$ is equivalent to $e\cdot\ker\big(\phi_v \mid W_F\big) = 0$ and hence also to the non-vanishing of $e\cdot P_v$.

Taken together, these facts imply that $e\cdot H^2(\mathcal{O}_{F,S(F')}, V) = 0$ if and only if one has both $e\cdot H^2(\mathcal{O}_{F,S(F)}, V) = 0$ (or equivalently, $e\cdot e_F \neq 0$) and also $e\cdot P_v \neq 0$ for each $v$ in $\Sigma$, as claimed. □

**Lemma 4.12.** *Fix a field $F$ in $\Omega(\mathcal{K}/K)$, a place $v$ of $K$ outside $S(F)$ and a primitive idempotent $e$ of $\zeta(A[\mathcal{G}_F])$ such that $e\cdot P_v \neq 0$. Write $C_v$ for the complex $\mathsf{R}\Gamma(K_v^{\mathrm{ur}}/K_v, V^*(1)_F)^*[-2]$ and set $W_F := \mathrm{Hom}_{\mathcal{Q}}(V^*(1)_F, \mathcal{Q})$. Then $C_v$ is represented by the complex $W_F \xrightarrow{1-\sigma_v} W_F$,* *where the first term is placed in degree one. In addition, the complex $e\cdot C_v$ is acyclic and $(e\cdot P_v,0)$ is equal to the image of $(e,0)$ under the composite isomorphism*

$$\begin{aligned}(\zeta(A[\mathcal{G}_F])e,0) &\cong e\cdot\big(\mathrm{d}^{\diamond}_{A[\mathcal{G}_F]}(W_F)\otimes \mathrm{d}^{\diamond}_{A[\mathcal{G}_F]}(W_F)^{-1}\big)\\ &\cong e\cdot \mathrm{d}_{A[\mathcal{G}_F]}(C_v)^{-1}\\ &= \mathrm{d}_{A[\mathcal{G}_F]e}(e\cdot C_v)^{-1}\\ &\cong \mathrm{d}^{\diamond}_{A[\mathcal{G}_F]e}(0)\otimes \mathrm{d}^{\diamond}_{A[\mathcal{G}_F]e}(0)^{-1}\\ &= (\zeta(A[\mathcal{G}_F])e,0).\end{aligned}$$

*Here the first isomorphism is induced by the canonical identification*

$$\mathrm{d}^{\diamond}_{A[\mathcal{G}_F]}(W_F)\otimes \mathrm{d}^{\diamond}_{A[\mathcal{G}_F]}(W_F)^{-1}\cong(\zeta(A[\mathcal{G}_F]),0),$$

*the second by the identification* [13, (5.3.1)] *(with $\mathcal{A}$ replaced by $A[\mathcal{G}_F]$) and the third by* [13, Prop. 5.17(i)] *(with $A$ replaced by $A[\mathcal{G}_F]e$) and the acyclicity of $e\cdot C_v$.*

*Proof.* It is clear that $C_v$ is represented by the given complex $W_F \xrightarrow{1-\sigma_v} W_F$ and hence that $e\cdot C_v$ is acyclic whenever $e\cdot P_v\neq 0$.

The remaining assertion is then verified by a straightforward, and explicit, computation (following [6, Lem. 10]). The key point is that, since $W_F=\mathrm{Hom}_{\mathcal{Q}}(V^*(1)_F,\mathcal{Q})$ is a finitely generated free $\mathcal{Q}[\mathcal{G}_F]$-module, the same argument as in [13, Lem. 3.23] implies that

$$\mathrm{Nrd}_{A[\mathcal{G}_F]}\big(1-\sigma_v\mid W_F\big)=\mathrm{Nrd}_{A[\mathcal{G}_F]}\big(1-\sigma_v\mid V^*(1)_F\big)^{\#}=P_v.$$

□

4.3. **The proof of Theorem 2.19.** Following Proposition 4.6, we can fix a basis element $\eta\in\mathrm{VS}(T,\mathcal{K})$ of the $\xi(\mathcal{A}[[\mathrm{Gal}(\mathcal{K}/K)]])$-module $\mathrm{VS}(T,\mathcal{K})$ and then use the homomorphism

$$\Theta_{T,\mathcal{K}}:\mathrm{VS}(T,\mathcal{K})\to\mathrm{ES}_r(T,\mathcal{K})$$

constructed in Theorem 4.7 to define a system $\varepsilon:=\Theta_{T,\mathcal{K}}(\eta)\in\mathrm{ES}_r(T,\mathcal{K})$. It suffices to show that this system has the properties in claims (i) and (ii) of Theorem 2.19.

In addition, the fact that $\varepsilon$ has the property in claim (i) follows directly from the argument used to verify the final assertion of Theorem 4.7.

Turning to claim (ii) we note that Lemma 4.1(ii) combines with the isomorphism (4.2.1) for $E=L$ to imply the existence of an isomorphism of $\mathcal{A}[\mathcal{G}_L]$-modules

$$H^1(C_{L,S(L)}(T))\cong H^2(\mathcal{O}_{L,S(L)},T)\oplus\mathcal{A}[\mathcal{G}_L]^r=:X. \tag{4.3.1}$$

This isomorphism implies that the block matrix $\big(0_{d_L,r}\mid M_L\big)$ that occurs in Lemma 4.10(iv) constitutes a free presentation $\Pi_{T,L}$ of $X$ and that (in terms of the notation introduced at the beginning of §4.2.3) the $\mathcal{A}[\mathcal{G}_L]$-module $\mathrm{cok}(\theta_{M_L})$ is isomorphic to $H^2(\mathcal{O}_{L,S(L)},T)$.

Thus, since the choice of $\eta$ implies the element $x_\eta$ in (4.2.6) (with $F'$ replaced by $L$) is a unit of $\xi(\mathcal{A}[\mathcal{G}_L])$, the general result of [13, Prop. 4.21(ii)] applies to imply both that

$$\begin{aligned}&\xi(\mathcal{A}[\mathcal{G}_L])\cdot\{(\wedge_{i=1}^{i=r}\varphi_i)(\varepsilon_L):\varphi_i\in\mathrm{Hom}_{\mathcal{A}[\mathcal{G}_L]}(H^1(\mathcal{O}_{L,S(L)},T),\mathcal{A}[\mathcal{G}_L])\}\\ &=\xi(\mathcal{A}[\mathcal{G}_L])\cdot\{(\wedge_{i=1}^{i=r}\varphi_i)(\varepsilon_{M_L}):\varphi_i\in\mathrm{Hom}_{\mathcal{A}[\mathcal{G}_L]}(H^1(\mathcal{O}_{L,S(L)},T),\mathcal{A}[\mathcal{G}_L])\}\\ &=\mathrm{Fit}^r_{\mathcal{A}[\mathcal{G}_L]}\big(\Pi_{T,L}\big)\end{aligned} \tag{4.3.2}$$

and, for every subset $\{\varphi_i\}_{1\le i\le r}$ of $\mathrm{Hom}_{\mathcal{A}[\mathcal{G}_L]}(H^1(\mathcal{O}_{L,S(L)},T),\mathcal{A}[\mathcal{G}_L])$, that

$$(\wedge_{i=1}^{i=r}\varphi_i)(\varepsilon_L) = x_\eta\cdot(\wedge_{i=1}^{i=r}\varphi_i)(\varepsilon_{M_L}) \in \mathrm{Fit}^0_{\mathcal{A}[\mathcal{G}_L]}\big(H^2(\mathcal{O}_{L,S(L)},T)\big).$$

This completes the proof of Theorem 2.19.

4.4. **The proof of Theorem 2.22.** Claim (i) is a direct consequence of the distribution relation (2.1.1) and the fact that $S(L)=S(L_0)$ for all fields $L$ in $\Omega(\mathcal{L}/L_0)$.

However, in order to prove the remaining assertions of Theorem 2.22 we must first refine the construction of Lemma 4.10.

In the rest of this section we shall assume that $\mathcal{A}=\mathcal{O}$. We also set $R_\mathcal{L} := \mathcal{O}[[\mathcal{L}/K]]$ and $R_L := \mathcal{O}[\mathcal{G}_L]$ for each $L$ in $\Omega(\mathcal{L}/K)$.

4.4.1. For each pair of fields $L$ and $L'$ in $\Omega(\mathcal{L}/L_0)$ with $L\subseteq L'$ the isomorphism

$$R_L\otimes^{\mathsf{L}}_{R_{L'}} C_{L',S(L_0)}(T)\cong C_{L,S(L_0)}(T) \tag{4.4.1}$$

coming from Lemma 4.1(iv) induces a surjective homomorphism

$$H^1(C_{L',S(L_0)}(T))\to H^1(C_{L,S(L_0)}(T)). \tag{4.4.2}$$

The isomorphisms (4.3.1) can be chosen to be compatible with these homomorphisms as $L$ varies and hence combine to induce an isomorphism of $R_\mathcal{L}$-modules

$$\varprojlim_{L\in\Omega(\mathcal{L}/L_0)} H^1(C_{L,S(L)}(T))\cong R_\mathcal{L}^r\oplus H^2(\mathcal{O}_\mathcal{L},T) \tag{4.4.3}$$

where the inverse limit is taken with respect to the homomorphisms (4.4.2).

**Lemma 4.13.** *There exists an exact sequence of $R_\mathcal{L}$-modules*

$$0\to H^1(\mathcal{O}_\mathcal{L},T)\xrightarrow{\iota_\mathcal{L}} P_\mathcal{L}\xrightarrow{\theta_\mathcal{L}} P_\mathcal{L}\xrightarrow{\pi_\mathcal{L}}\varprojlim_{L\in\Omega(\mathcal{L}/L_0)} H^1(C_{L,S(L)}(T))\to 0 \tag{4.4.4}$$

*that has all of the following properties.*

(i) *The $R_\mathcal{L}$-module $P_\mathcal{L}$ is free of (finite) rank $d$ with $d>r$.*

(ii) *There exists an ordered basis $\{b_{i,\mathcal{L}}\}_{1\le i\le d}$ of $P_\mathcal{L}$ with the property that the composite of $\pi_\mathcal{L}$ and the isomorphism (4.4.3) sends $b_{i,\mathcal{L}}$ to the $i$-th element of the standard basis of $R_\mathcal{L}^r$ if $1\le i\le r$ and to an element of $H^2(\mathcal{O}_\mathcal{L},T)$ otherwise.*

(iii) *For each field $L$ in $\Omega(\mathcal{L}/L_0)$, with $\Gamma_L := \mathrm{Gal}(\mathcal{L}/L)$, there exists an exact sequence of the form (4.2.5) with the following properties: $P_L = H_0(\Gamma_L,P_\mathcal{L})$ (so $d_L=d$), $\theta_L = H_0(\Gamma_L,\theta_\mathcal{L})$, $\pi_L$ is the map induced by (taking $\Gamma_L$-coinvariants of) the surjective composite homomorphism*

$$P_\mathcal{L}\xrightarrow{\pi_\mathcal{L}}\varprojlim_{L'\in\Omega(\mathcal{L}/L_0)} H^1(C_{L',S(L')}(T))\to H^1(C_{L,S(L)}(T))$$

*where the last map is induced by the surjections (4.4.2), and the image of $\{b_{i,\mathcal{L}}\}_{1\le i\le d}$ in $P_L$ is an $R_L$-basis of $P_L$ with all of the properties required by Lemma 4.10(ii).*

*Proof.* At the outset we fix a surjective homomorphism of $R_\mathcal{L}$-modules

$$\pi_\mathcal{L} : P_\mathcal{L}\to\varprojlim_{L\in\Omega(\mathcal{L}/L_0)} H^1(C_{L,S(L)}(T))$$

that has the properties in claims (i) and (ii).

For each $L$ in $\Omega(\mathcal{L}/L_0)$ we then set $P_L := H_0(\mathrm{Gal}(\mathcal{L}/L), P_{\mathcal{L}})$ and define $\pi_L$ to be the displayed composite homomorphism in claim (iii).

We now set $S_0 := S(L_0)$ and choose an ordered set of fields $\{L_i : i \in \mathbb{N}\}$ in $\Omega(\mathcal{L}/L_0)$ that is cofinal with respect to inclusion and for each $n$ abbreviate $C_{L_n,S_0}(T), R_{L_n}, P_{L_n}$ and $\pi_{L_n}$ to $C_n(T), R_n, P_n$ and $\pi_n$. We then consider the diagrams

$$\begin{array}{ccccccccccc} 0 & \to & H^1(\mathcal{O}_{L_{n+1},S_0}, T) & \xrightarrow{\iota_{n+1}} & P_{n+1} & \xrightarrow{\theta_{n+1}} & P_{n+1} & \xrightarrow{\pi_{n+1}} & H^1(C_{n+1}(T)) & \to & 0 \\ & & \kappa\big\downarrow & & \varrho'\big\downarrow & & \big\downarrow\varrho & & \big\downarrow\lambda & & \\ 0 & \to & H^1(\mathcal{O}_{L_n,S_0}, T) & \xrightarrow{\iota_n} & P_n & \xrightarrow{\theta_n} & P_n & \xrightarrow{\pi_n} & H^1(C_n(T)) & \to & 0. \end{array} \tag{4.4.5}$$

Here the rows are the respective sequences constructed in Lemma 4.10 (with $\pi_{n+1}$ and $\pi_n$ taken to be the maps specified above), $\varrho$ is the natural projection map, $\lambda$ the relevant case of (4.4.2) and $\kappa$ the corestriction map. Finally, $\varrho'$ is a homomorphism of $R_{n+1}$-modules chosen so that the second square commutes and the resulting morphism of complexes represented by the pair $(\varrho', \varrho)$ induces the isomorphism (4.4.1) for the extension $L_{n+1}/L_n$.

One checks that this construction guarantees that the diagram commutes and also that $\varrho'$ induces an isomorphism of $R_n$-modules $H_0(\mathrm{Gal}(L_{n+1}/L_n), P_{n+1}) \cong P_n$.

Hence, if by induction we fix a compatible family of such diagrams for all $n$, then we can pass to the inverse limit over $n$ of the diagrams to obtain an exact sequence of the form (4.4.4). We note that exactness is preserved by this limiting process since each module in the diagram (4.4.5) is compact, and also that the resulting morphism $\pi_{\mathcal{L}}$ coincides with the morphism fixed at the start of this argument.

For each $L$ in $\Omega(\mathcal{L}/L_0)$ we now choose $n$ so that $L \subseteq L_n$. Then the isomorphism (4.4.1) with $L' = L_n$ implies that one obtains an exact sequence of the form (4.2.5) for the field $L$ by taking $\mathrm{Gal}(L_n/L)$-coinvariants of the complex $P_n \xrightarrow{\theta_n} P_n$ fixed above.

With this construction it is also clear that the image of $\{b_{i,\mathcal{L}}\}_{1\le i\le d}$ in $P_L$ is an $R_L$-basis of $P_L$ with the properties stated in Lemma 4.10(ii). □

To proceed we note that the exactness of the sequence (4.4.4) combines with Lemma 4.13(ii) to imply that the matrix of the endomorphism $\theta_{\mathcal{L}}$ with respect to the basis $\{b_{i,\mathcal{L}}\}_{1\le i\le d}$ of $P_{\mathcal{L}}$ is a block matrix of the form $\big(0_{d,r} \mid M_{\mathcal{L}}\big)$. Here $M_{\mathcal{L}}$ belongs to $\mathrm{M}_{d,d-r}(R_{\mathcal{L}})$ and is such that for each $L$ in $\Omega(\mathcal{L}/L_0)$ its projection to $\mathrm{M}_{d,d-r}(\mathcal{O}[\mathcal{G}_L])$ is the matrix $M_L$ that occurs in Lemma 4.10(iv).

The sequence (4.4.4) therefore combines with the isomorphism (4.4.3) to imply the $R_{\mathcal{L}}$-module $H^2(\mathcal{O}_{\mathcal{L}}, T)$ is torsion if and only if there exists $M'_{\mathcal{L}} \in \mathrm{M}_{d,r}(R_{\mathcal{L}})$ such that

$$\big(M'_{\mathcal{L}} \mid M_{\mathcal{L}}\big) \in \mathrm{GL}_d(Q(R_{\mathcal{L}})). \tag{4.4.6}$$

Finally we note that each column of $M'_{\mathcal{L}}$ corresponds (via the fixed basis of $P_{\mathcal{L}}$) to an element of $\mathrm{Hom}_{R_{\mathcal{L}}}(P_{\mathcal{L}}, R_{\mathcal{L}})$ and hence, by restriction through $\iota_{\mathcal{L}}$, to an element of $\mathrm{Hom}_{R_{\mathcal{L}}}(H^1(\mathcal{O}_{\mathcal{L}}, T), R_{\mathcal{L}})$.

The following result describes the set of homomorphisms that can arise in this way. This result uses the inverse limit $\mathrm{Hom}^*_{R_{\mathcal{L}}}\big(H^1(\mathcal{O}_{\mathcal{L}}, T), R_{\mathcal{L}}\big)$ defined in (2.2.6).

**Lemma 4.14.** *The image of the restriction map*

$$\mathrm{Hom}_{R_{\mathcal{L}}}(P_{\mathcal{L}}, R_{\mathcal{L}}) \to \mathrm{Hom}_{R_{\mathcal{L}}}(H^1(\mathcal{O}_{\mathcal{L}}, T), R_{\mathcal{L}})$$

*that is induced by $\iota_{\mathcal{L}}$ coincides with the image of the natural map*

$$\mathrm{Hom}^*_{R_{\mathcal{L}}}\big(H^1(\mathcal{O}_{\mathcal{L}},T),R_{\mathcal{L}}\big) \to \mathrm{Hom}_{R_{\mathcal{L}}}(H^1(\mathcal{O}_{\mathcal{L}},T),R_{\mathcal{L}}).$$

*Proof.* The exact sequence (4.4.4) gives rise to an exact commutative diagram

$$\begin{array}{ccccc} \mathrm{Hom}_{R_{\mathcal{L}}}(P_{\mathcal{L}},R_{\mathcal{L}}) & \xrightarrow{\iota^*_{\mathcal{L}}} & \mathrm{Hom}_{R_{\mathcal{L}}}(H^1(\mathcal{O}_{\mathcal{L}},T),R_{\mathcal{L}}) & \to & \mathrm{Ext}^1_{R_{\mathcal{L}}}(\mathrm{im}(\theta_{\mathcal{L}}),R_{\mathcal{L}}) \\ \alpha\Big\downarrow\cong & & \beta\Big\uparrow & & \\ \varprojlim_L \mathrm{Hom}_{R_L}(P_L,R_L) & \xrightarrow{(\iota^*_L)_L} & \mathrm{Hom}^*_{R_{\mathcal{L}}}\big(H^1(\mathcal{O}_{\mathcal{L}},T),R_{\mathcal{L}}\big). & & \end{array} \tag{4.4.7}$$

Here the inverse limit runs over all $L$ in $\Omega(\mathcal{L}/L_0)$ and is taken with respect to the projection maps that are induced by the fact each $R_L$-module $P_L = H_0(\Gamma_L, P_{\mathcal{L}})$ is free of rank $d$, $\iota^*_{\mathcal{L}}$ denotes the restriction map induced by $\iota_{\mathcal{L}}$, $\alpha$ the natural isomorphism (which exists since $P_{\mathcal{L}} \cong \varprojlim_{L\in\Omega(\mathcal{L}/L_0)} P_L$ is a free $R_{\mathcal{L}}$-module), $(\iota^*_L)_L$ the inverse limit of the homomorphisms

$$\iota^*_L : \mathrm{Hom}_{R_L}(P_L,R_L) \to \mathrm{Hom}_{R_L}(H^1(\mathcal{O}_{L,S(L)},T),R_L)$$

that are induced by restriction through $\iota_L$ and $\beta$ is the natural map.

To prove the claimed result it is therefore enough to show that $(\iota^*_L)_L$ is surjective. But, for each $L$, the image of the endomorphism $\theta_L$ in the sequence (4.2.5) is torsion-free and so, since $R_L$ is Gorenstein, the group $\mathrm{Ext}^1_{R_L}(\mathrm{im}(\theta_L),R_L)$ vanishes (cf. Remark 2.2). From the exactness of (4.2.5) one can therefore deduce that $\iota^*_L$ is surjective and this surjectivity is then preserved upon passing to the inverse limit over $L$ since each of the modules $\ker(\iota^*_L)$ is compact. □

4.4.2. To prove claim (ii) of Theorem 2.22 we assume until further notice that $\mathrm{Gal}(\mathcal{L}/K)$ has rank one. In this case we also assume, as we may, that $\mathrm{Gal}(\mathcal{L}/L_0)$ is central in $\mathrm{Gal}(\mathcal{L}/K)$.

Since $\mathrm{Gal}(\mathcal{L}/K)$ has rank one the central conductor formula of Nickel [42, Th. 3.5] implies (via [42, Cor. 4.1]) that the group $\mathrm{Ext}^1_{R_{\mathcal{L}}}(\mathrm{im}(\theta_{\mathcal{L}}),R_{\mathcal{L}})$ is annihilated by a power of $p$.

From the diagram (4.4.7) it therefore follows that the condition (4.4.6) is satisfied by a matrix $M'_{\mathcal{L}}$ in $\mathrm{M}_{d,r}(R_{\mathcal{L}})$ if and only if it is satisfied by such a matrix with the property that, for each $i$ with $1 \le i \le r$, its $i$-th column corresponds to an element of $\mathrm{Hom}_{R_{\mathcal{L}}}(P_{\mathcal{L}},R_{\mathcal{L}})$ whose image under $\iota^*_{\mathcal{L}}$ is equal to $\beta(\varphi_i)$ for some $\varphi_i = (\varphi_{i,L})_L$ in $\mathrm{Hom}^*_{R_{\mathcal{L}}}\big(H^1(\mathcal{O}_{\mathcal{L}},T),R_{\mathcal{L}}\big)$.

In addition, in this case the algebra $Q(R_{\mathcal{L}})$ is semisimple and so (4.4.6) is satisfied by any such matrix $M'_{\mathcal{L}}$ if and only if $\mathrm{Nrd}_{Q(R_{\mathcal{L}})}\big((M'_{\mathcal{L}} \mid M_{\mathcal{L}})\big)$ belongs to $\zeta(Q(R_{\mathcal{L}}))^\times$.

To complete the proof of Theorem 2.22(ii) it is thus enough to prove the existence of a unit $u$ of $\xi(R_{\mathcal{L}})$ for which one has

$$\big(\wedge_{i=1}^{i=r}\varphi_i\big)(\varepsilon_{\mathcal{L}}) = u \cdot \mathrm{Nrd}_{Q(R_{\mathcal{L}})}\big((M'_{\mathcal{L}} \mid M_{\mathcal{L}})\big). \tag{4.4.8}$$

To prove this we need the following result.

**Lemma 4.15.** $\mathrm{Nrd}_{Q(R_{\mathcal{L}})}\big((M'_{\mathcal{L}} \mid M_{\mathcal{L}})\big)$ *belongs to* $\xi(R_{\mathcal{L}})$ *and is equal to*

$$(\mathrm{Nrd}_{\mathcal{Q}[\mathcal{G}_L]}\big((M'_L \mid M_L)\big)_{L\in\Omega(\mathcal{L}/L_0)},$$

*where* $M'_L$ *denotes the image of* $M'_{\mathcal{L}}$ *in* $\mathrm{M}_{d,r}(R_L)$.

*Proof.* We use the explicit description of $\mathrm{Nrd}_{Q(R_{\mathcal{L}})}$ given by Ritter and Weiss in [44]. To do so we note that, as $\mathcal{G} := \mathrm{Gal}(\mathcal{L}/K)$ has rank one, it has a finite closed normal subgroup $\mathcal{H}$ for which the quotient group $\Gamma := \mathcal{G}/\mathcal{H}$ is topologically isomorphic to $\mathbb{Z}_p$. We set $\Lambda := \mathcal{O}[[\Gamma]]$ and for each $L$ in $\Omega(\mathcal{L}/L_0)$ and $\chi$ in $\widehat{\mathcal{G}_L}$ we use the homomorphism

$$j_\chi : \zeta(Q(R_{\mathcal{L}})) \to \mathbb{Q}^c \otimes_{\mathcal{O}} Q(\Lambda)$$

that is defined in [44, Prop. 6].

We now set $\Phi := \big(M'_{\mathcal{L}} \mid M_{\mathcal{L}}\big)$. Then for every $L$ in $\Omega(\mathcal{L}/L_0)$ and $\chi$ in $\widehat{\mathcal{G}_L}$ the element $j_\chi(\mathrm{Nrd}_{Q(R_{\mathcal{L}})}(\Phi))$ belongs to $\mathbb{Q}_p^c \otimes_{\mathcal{O}} \zeta(\Lambda)$ (this is observed, for example, in the result [40, Th. 6.4] of Nickel). This fact combines with the proof of [44, Th. 7] to imply $\mathrm{Nrd}_{Q(R_{\mathcal{L}})}(\Phi)$ belongs to $\mathbb{Q}_p^c \otimes_{\mathcal{O}} \zeta(R_{\mathcal{L}})$ and so it is enough to show that for every $L$ in $\Omega(\mathcal{L}/L_0)$ the natural projection map $\varrho_L : \mathbb{Q}_p^c \otimes_{\mathcal{O}} \zeta(R_{\mathcal{L}}) \to \zeta(\mathbb{Q}_p^c[\mathcal{G}_L])$ sends $\mathrm{Nrd}_{Q(R_{\mathcal{L}})}(\Phi)$ to $\mathrm{Nrd}_{\mathcal{Q}[\mathcal{G}_L]}(\Phi_L)$ with

$$\Phi_L := \big(M'_L \mid M_L\big).$$

To prove this we note that, for each $\chi$ in $\widehat{\mathcal{G}_L}$, the homomorphism $j_\chi$ maps $\mathbb{Q}_p^c \otimes_{\mathcal{O}} \zeta(R_{\mathcal{L}})$ to $\mathbb{Q}_p^c \otimes_{\mathcal{O}} \zeta(\Lambda)$ and there exists a commutative diagram

$$\begin{array}{ccc}
\mathbb{Q}_p^c \otimes_{\mathcal{O}} \zeta(R_{\mathcal{L}}) & \xrightarrow{j_\chi} & \mathbb{Q}_p^c \otimes_{\mathcal{O}} \zeta(\Lambda) \\
{\scriptstyle \varrho_L}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{aug}_\Gamma} \\
\zeta(\mathbb{Q}_p^c[\mathcal{G}_L]) & \xrightarrow{\theta_\chi} & \mathbb{Q}_p^c.
\end{array}$$

Here $\mathrm{aug}_\Gamma$ is induced by the projection map $\Lambda \to \mathcal{O}$ and $\theta_\chi$ is the projection induced by multiplication by the idempotent $e_\chi$, and the commutativity of the diagram follows, for example, from the argument used at the end of the proof of [40, Th. 6.4] to establish an equality '$\pi(\lambda) = \overline{\lambda}$'. Given this diagram, and the fact that $\bigcap_{\chi\in\widehat{\mathcal{G}_L}} \ker(\theta_\chi)$ vanishes, the required result follows directly from the equality

$$\mathrm{aug}_\Gamma(j_\chi(\mathrm{Nrd}_{Q(R_{\mathcal{L}})}(\Phi))) = \theta_\chi\big(\mathrm{Nrd}_{\mathcal{Q}[\mathcal{G}_L]}(\Phi_L)\big)$$

that is proved for each $\chi \in \widehat{\mathcal{G}_L}$ in [40, (8)]. □

Next we note that, for each field $L$ in $\Omega(\mathcal{L}/L_0)$, the equality (4.2.8) combines with our choice of homomorphisms $\varphi_i = (\varphi_{i,L})_L$ to imply that

$$\mathrm{Nrd}_{\mathcal{Q}[\mathcal{G}_L]}\big(\big(M'_L \mid M_L\big)\big) = (\wedge_{i=1}^{i=r}\varphi_{i,L})(\varepsilon_{M_L}). \tag{4.4.9}$$

In addition, by an explicit comparison of the definitions of the elements $\varepsilon_L$ and $\varepsilon_{M_L}$ for each $L$ in $\Omega(\mathcal{L}/L_0)$, one derives the existence of a (unique) unit $u$ of $\xi(R_{\mathcal{L}})$ for which one has $\varepsilon_L = u \cdot \varepsilon_{M_L}$ for all such $L$.

Hence, since $\big(\wedge_{i=1}^{i=r}\varphi_i\big)(\varepsilon_{\mathcal{L}})$ is equal to $\big((\wedge_{i=1}^{i=r}\varphi_{i,L})(\varepsilon_L)\big)_{L\in\Omega(\mathcal{L}/L_0)}$, the equality (4.4.9) combines with Lemma 4.15 to imply that this element $u$ validates the required equality (4.4.8).

This completes the proof of Theorem 2.22(ii).

4.4.3. To prove claim (iii) of Theorem 2.22 we no longer assume that $\mathrm{Gal}(\mathcal{L}/K)$ has rank one but we do assume to be given a subset $\{\varphi_i\}_{1\le i\le r}$ of $\mathrm{Hom}^*_{R_\mathcal{L}}\big(H^1(\mathcal{O}_\mathcal{L},T),R_\mathcal{L}\big)$ for which $\big(\wedge_{i=1}^{i=r}\varphi_i\big)(\varepsilon_\mathcal{L})$ is a non-zero divisor in the ideal $\delta(R_\mathcal{L})=\varprojlim_{L\in\Omega(\mathcal{L}/L_0)}\delta(R_L)$ of $\zeta(R_\mathcal{L})$.

Then the square of $\big(\wedge_{i=1}^{i=r}\varphi_i\big)(\varepsilon_\mathcal{L})$ is a non-zero divisor in $\zeta(R_\mathcal{L})$ and so it is enough to prove that this element annihilates $H^2(\mathcal{O}_\mathcal{L},T)$.

To show this we note the given assumption on the homomorphisms $\varphi_i=(\varphi_{i,L})_{L\in\Omega(\mathcal{L}/L_0)}$ implies that for all $L$ in $\Omega(\mathcal{L}/L_0)$ one has $\big(\wedge_{i=1}^{i=r}\varphi_{i,L}\big)(\varepsilon_L)\in\delta(R_L)$. For each such $L$, Theorem 2.19(ii) therefore implies that

$$\big(\big(\wedge_{i=1}^{i=r}\varphi_{i,L}\big)(\varepsilon_L)\big)^2\in\delta(R_L)\cdot\mathrm{Fit}^0_{R_L}(H^2(\mathcal{O}_{L,S(L)},T))\subseteq\mathrm{Ann}_{R_L}(H^2(\mathcal{O}_{L,S(L)},T)),$$

where the inclusion follows from the general result of [13, Th. 3.20(iii)].

This containment in turn implies that the square of $\big(\wedge_{i=1}^{i=r}\varphi_i\big)(\varepsilon_\mathcal{L})$ annihilates the module $H^2(\mathcal{O}_\mathcal{L},T)=\varprojlim_{L\in\Omega(\mathcal{L}/L_0)}H^2(\mathcal{O}_{L,S(L)},T)$, as required to prove claim (iii).

4.4.4. Turning now to claim (iv) of Theorem 2.22 we assume $\mathcal{L}/K$ is abelian so that $Q(R_\mathcal{L})$ is a finite product of fields.

In this case the exact sequence (4.4.4) combines with the isomorphism (4.4.3) to imply that $H^1(\mathcal{O}_\mathcal{L},T)$ spans over each field component of $Q(R_\mathcal{L})$ a vector space of dimension at least $r$ and that this dimension is equal to $r$ in each component if and only if $H^2(\mathcal{O}_\mathcal{L},T)$ is a torsion $R_L$-module. This implies, in particular, that $H^2(\mathcal{O}_\mathcal{L},T)$ is a torsion $R_\mathcal{L}$-module if and only if the $Q(R_\mathcal{L})$-module spanned by $\bigwedge^r_{R_\mathcal{L}}H^1(\mathcal{O}_\mathcal{L},T)$ is free of rank one.

Thus, since the earlier argument showed that $H^2(\mathcal{O}_\mathcal{L},T)$ is a torsion $R_\mathcal{L}$-module if $\varepsilon_\mathcal{L}$ satisfies the condition stated in claim (ii), respectively in claim (iii), it is clear that the condition in (iv)(a) implies the condition in (iv)(b). Since it is obvious that (iv)(b) implies the condition in (iv)(c) it is therefore enough to show that the latter condition implies that $\varepsilon_\mathcal{L}$ satisfies the condition in either claim (ii) or claim (iii). To do this we note that the exact sequence (4.4.4) implies the existence of a unit $u$ of $R_\mathcal{L}$ such that

$$\varepsilon_\mathcal{L}=u\cdot\big(\wedge_{r<i\le d}(b^*_{i,\mathcal{L}}\circ\theta_\mathcal{L})(\wedge_{i=1}^{i=d}b_{i,\mathcal{L}})\big).$$

In particular, if this element is not annihilated by any non zero-divisor of $R_\mathcal{L}$, then for each field component of $Q(R_\mathcal{L})$ there exists an $r$-tuple $\{i_j\}_{1\le j\le r}$ of integers with $1\le i_j\le d$ such that the corresponding component of $(\wedge_{1\le j\le r}(b^*_{i_j,\mathcal{L}}))(\varepsilon_\mathcal{L})$ is non-zero.

By using this fact it is then easy to construct a subset $\{\varphi_i\}_{1\le i\le r}$ of $\mathrm{Hom}_{R_\mathcal{L}}(P_\mathcal{L},R_\mathcal{L})$, and hence via the diagram (4.4.7), of $\mathrm{Hom}^*_{R_\mathcal{L}}(H^1(\mathcal{O}_\mathcal{L},T),R_\mathcal{L})$ such that $\big(\wedge_{i=1}^{i=r}\varphi_i\big)(\varepsilon_\mathcal{L})$ is a non-zero divisor in $R_\mathcal{L}=\delta(R_\mathcal{L})$, as required to verify the condition in (ii)(a).

This completes the proof of Theorem 2.22.

## 5. Cyclotomic non-commutative Euler systems

In this final section we shall use the techniques developed above to construct (unconditionally) an extension of the classical Euler system of cyclotomic units to the setting of general totally real Galois extensions of $\mathbb{Q}$ and verify that this extended system has all of the properties that are stated in Theorem B.

We fix an odd prime $p$ and write $A_p$ for the pro-$p$ completion of an abelian group $A$. We also fix an isomorphism of fields $\mathbb{C} \cong \mathbb{C}_p$ and use it to identity (without further explicit comment) the set $\mathrm{Ir}(\Gamma)$ of irreducible complex characters of a finite group $\Gamma$ with the corresponding set $\mathrm{Ir}_p(\Gamma)$ of irreducible $\mathbb{C}_p$-valued characters of $\Gamma$.

## 5.1. **The construction.**

5.1.1. For any finite non-empty set of places $\Sigma$ of $\mathbb{Q}$ and any number field $F$ we write $\Sigma_F$ for the set of places of $F$ lying above those in $\Sigma$, $Y_{F,\Sigma}$ for the free abelian group on the set $\Sigma_F$ and $X_{F,\Sigma}$ for the submodule of $Y_{F,\Sigma}$ comprising elements whose coefficients sum to zero.

If $\Sigma$ contains $\infty$, then we write $\mathcal{O}_{F,\Sigma}$ for the subring of $F$ comprising elements integral at all places outside $\Sigma_F$. (If $\Sigma = \{\infty\}$, then we usually abbreviate $\mathcal{O}_{F,\Sigma}$ to $\mathcal{O}_F$.)

For any such $\Sigma$ the $\Sigma$-relative Selmer group $\mathrm{Sel}_\Sigma(F)$ for $\mathbb{G}_m$ over $F$ is defined in [8, §2.1] to be the cokernel of the homomorphism

$$\prod\nolimits_{w\notin\Sigma_F}\mathbb{Z} \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(F^\times, \mathbb{Z}),$$

where $w$ runs over all places of $F$ that do not belong to $\Sigma_F$ and the arrow sends $(x_w)_w$ to the map $(u \mapsto \sum_w \mathrm{ord}_w(u)x_w)$ with $\mathrm{ord}_w$ the normalised additive valuation at $w$. (This group is a natural analogue for $\mathbb{G}_m$ of the integral Selmer groups of abelian varieties that are defined by Mazur and Tate in [35].)

We recall that $\mathrm{Sel}_\Sigma(F)_p$ lies in a canonical exact sequence

$$0 \to \mathrm{Cl}(\mathcal{O}_{F,\Sigma})_p^\vee \to \mathrm{Sel}_\Sigma(F)_p \to \mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{O}_{F,\Sigma,p}^\times, \mathbb{Z}_p) \to 0 \tag{5.1.1}$$

and has a subquotient that is canonically isomorphic to $\mathrm{Cl}(\mathcal{O}_F)_p^\vee$ (cf. [8, Prop. 2.2]).

If $F$ is Galois over $\mathbb{Q}$ (so that $F \in \Omega(\mathbb{Q}^c/\mathbb{Q})$), then we set $\mathcal{G}_F := \mathrm{Gal}(F/\mathbb{Q})$ and

$$C_F := C_{F,S(F)}(\mathbb{Z}_p(1)),$$

where the latter complex is as defined in §4.1.1, and also $C_F^* := \mathsf{R}\,\mathrm{Hom}_{\mathbb{Z}_p}(C_F, \mathbb{Z}_p)$.

In this case, for each character $\chi$ in $\mathrm{Ir}(\mathcal{G}_F)$, and each integer $a$, we also write $L^a_{S(F)}(\chi, 0)$ for the coefficient of $z^a$ in the Laurent expansion of $L^a_{S(F)}(\chi, z)$ at $z = 0$.

We recall that $\check{\chi}$ denotes the contragredient of $\chi$.

**Lemma 5.1.** *For every finite Galois extension $F$ of $\mathbb{Q}$ the following claims are valid.*

(i) $H^0(\mathcal{O}_{F,S(F)}, \mathbb{Z}_p(1))$ *vanishes.*

(ii) $H^1(\mathcal{O}_{F,S(F)}, \mathbb{Z}_p(1)) = H^0(C_F)$ *identifies with* $\mathcal{O}^\times_{F,S(F),p}$.

(iii) *There exists a canonical exact sequence*

$$0 \to \mathrm{Cl}(\mathcal{O}_{F,S(F)})_p \to H^1(C_F) \to X_{F,S(F),p} \to 0.$$

(iv) $H^2(C_F^*[-2]) = H^2_c(\mathcal{O}_{F,S(F)}, \mathbb{Z}_p)$ *identifies with* $\mathrm{Sel}_{S(F)}(F)_p$.

(v) $Y_F(\mathbb{Z}_p) = Y_{F,S_\infty(F),p}$ *and if $F$ is totally real, then the idempotent $e_F = e_{F,\mathbb{Z}_p(1)}$ defined in (2.2.5) is equal to $\sum_\chi e_\chi$ where $\chi$ runs over all characters in* $\mathrm{Ir}(\mathcal{G}_F)$ *for which* $L^{\chi(1)}_{S(F)}(\check{\chi}, 0) \neq 0$.

*Proof.* Claim (i) is obvious and the existence of the identification in claim (ii) and exact sequence in claim (iii) are well-known, being respectively induced by Kummer theory and class field theory.

Since the complex $C_F^*[-2] = (\mathsf{R}\Gamma_c(\mathcal{O}_{F,S(F)}, \mathbb{Z}_p)^*[-2])^*[-2]$ is canonically isomorphic to $\mathsf{R}\Gamma_c(\mathcal{O}_{F,S(F)}, \mathbb{Z}_p)$ the identification in claim (iv) is proved by the argument of [8, Prop. 2.4(iii)].

The first assertion of claim (v) is obvious. To prove the second assertion, we note that the exact sequence in claim (iii) combines with that in Lemma 4.1(ii) to imply $e_F$ is equal to $\sum_\chi e_\chi$ where $\chi$ runs over all characters of $\mathcal{G}_F$ for which the natural map

$$e_\chi(\mathbb{C} \otimes X_{F,S(F)}) \to e_\chi(\mathbb{C} \otimes Y_{F,S_\infty(F)})$$

is bijective. In addition, if $F$ is totally real, then the $\mathcal{G}_F$-module $Y_{F,S_\infty(F)}$ is free of rank one and so one has $\mathrm{rr}_{\mathbb{C}[\mathcal{G}_F]e_\chi}(e_\chi(\mathbb{C} \otimes Y_{F,S_\infty(F)})) = \chi(1)$ (cf. [13, Rem. 2.5]). Given these observations, the second assertion of claim (v) follows directly from the formula

$$\mathrm{ord}_{z=0} L_{S(F)}(\check{\chi}, z) = \mathrm{rr}_{\mathbb{C}[\mathcal{G}_F]e_\chi}(e_\chi(\mathbb{C} \otimes X_{F,S(F)})) \tag{5.1.2}$$

that is proved, for example, in [47, Chap. I, Prop. 3.4]. □

**Remark 5.2.** If $\Sigma$ is any finite set of places of $F$ that contains all archimedean and $p$-adic places, then the same approach as above shows that $H^0(C_{F,\Sigma}(\mathbb{Z}_p(1))) = \mathcal{O}^\times_{F,\Sigma,p}$ and that there is a canonical exact sequence $0 \to \mathrm{Cl}(\mathcal{O}_{F,\Sigma})_p \to H^1(C_{F,\Sigma}(\mathbb{Z}_p(1))) \to X_{F,\Sigma,p} \to 0$.

5.1.2. Since $p$ is fixed we shall in the sequel set

$$\Lambda := \mathbb{Z}_p[[\mathrm{Gal}(\mathbb{Q}^{c,+}/\mathbb{Q})]] \ \text{ and } \ \Lambda^{\mathrm{ab}} := \mathbb{Z}_p[[\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab},+}/\mathbb{Q})]].$$

We also abbreviate the sets $\Omega(\mathbb{Q}^{c,+}/\mathbb{Q})$ and $\Omega(\mathbb{Q}^{\mathrm{ab},+}/\mathbb{Q})$ to $\Omega$ and $\Omega^{\mathrm{ab}}$ respectively.

For each field $F$ in $\Omega$ we assume that the constructions of reduced exterior powers, reduced Rubin lattices and reduced determinants over the group ring $\mathbb{Q}_p[\mathcal{G}_F]$ are normalized via the choice of data fixed in §2.1.1 (with $K = \mathbb{Q}$).

At the outset we check that the conditions of Theorem 2.19 (and Theorem 4.7) are satisfied with $K = \mathbb{Q}, \mathcal{K} = \mathbb{Q}^{c,+}$, $T = \mathbb{Z}_p(1)$ (so that $T^*(1) = \mathbb{Z}_p$) and $\mathcal{A} = \mathbb{Z}_p$.

Firstly, since $p$ is odd, for each $L$ in $\Omega$ the group $\mathcal{O}^\times_{L,S(L),p}$ is torsion-free and so Lemma 5.1(ii) implies condition (a) of Theorem 2.19 is satisfied. In addition, since $\mathcal{A} = \mathbb{Z}_p$ is local, condition (b) is clear and condition (c) is satisfied with $r := r_{\mathbb{Z}_p(1)}$ equal to 1 since $Y_{\mathbb{Q}}(\mathbb{Z}_p) = \mathbb{Z}_p$. Finally, condition (d) is true by the very definition of $\mathcal{K} = \mathbb{Q}^{c,+}$.

Next we fix a field embedding

$$\sigma : \mathbb{Q}^c \to \mathbb{C}. \tag{5.1.3}$$

Then for each $L$ in $\Omega$ the restriction of $\sigma$ gives an embedding $\sigma_L : L \to \mathbb{R}$. This embedding $\sigma_L$ defines a place $w_L$ in $S_\infty(L)$ that constitutes, via Lemma 5.1(v), a $\mathbb{Z}_p[\mathcal{G}_L]$-basis of $Y_L(\mathbb{Z}_p)$. In this way the fixed embedding $\sigma$ gives rise to a compatible family of isomorphisms of the form (4.2.1). By applying Theorem 4.7 with respect to this family of isomorphisms, we obtain canonical homomorphisms of $\xi(\Lambda)$-modules

$$\Theta := \Theta_{\mathbb{Z}_p(1),\mathbb{Q}^{c,+}} \ \text{ and } \ \Theta^{\mathrm{ab}} := \Theta_{\mathbb{Z}_p(1),\mathbb{Q}^{\mathrm{ab},+}}.$$

For each natural number $n$ we write $\zeta_n$ for the unique primitive $n$-th root of unity in $\mathbb{Q}^c$ that satisfies

$$\sigma(\zeta_n) = e^{2\pi i/n}. \tag{5.1.4}$$

(It is then clear that $\zeta_m = (\zeta_n)^{n/m}$ for all divisors $m$ of $n$.) For each field $L$ in $\Omega^{\mathrm{ab}}$, of conductor $f(L)$, we then define an element of $L^\times$ by setting

$$\epsilon_L := \mathrm{Norm}_{\mathbb{Q}(\zeta_{f(L)})/\mathbb{Q}}(1-\zeta_{f(L)}).$$

The following result will be deduced from the validity of the equivariant Tamagawa Number Conjecture for abelian fields, as proved by the first author and Greither in [7].

**Theorem 5.3.** *There exists a basis element $\eta^{\mathrm{ab}}$ of the $\Lambda^{\mathrm{ab}}$-module $\mathrm{VS}(\mathbb{Z}_p(1), \mathbb{Q}^{\mathrm{ab},+})$ with the property that for every $L$ in $\Omega^{\mathrm{ab}}$ one has*

$$\Theta_L(\eta^{\mathrm{ab}}) = \begin{cases} \epsilon_L, & \text{if } p \text{ ramifies in } L, \\ (\epsilon_L)^{1-\sigma_{p,L}}, & \text{if } p \text{ is unramified in } L, \end{cases}$$

*where $\Theta_L$ denotes the $L$-component of $\Theta^{\mathrm{ab}}$ and $\sigma_{p,L}$ the restriction of $\sigma_p$ to $L$.*

*Proof.* For $E$ in $\Omega^{\mathrm{ab}}$ we write $\vartheta_E$ for the composite morphism in $\mathcal{P}(\mathbb{C}_p[\mathcal{G}_E])$

$$\begin{aligned} \mathbb{C}_p \otimes_{\mathbb{Z}_p} \mathrm{d}_{\mathbb{Z}_p[\mathcal{G}_E]}(C_E) &\xrightarrow{\sim} \mathrm{d}^\diamond_{\mathbb{C}_p[\mathcal{G}_E]}(\mathbb{C}_p \otimes_{\mathbb{Z}} \mathcal{O}^\times_{E,S(E)}) \otimes \mathrm{d}^\diamond_{\mathbb{C}_p[\mathcal{G}_E]}(\mathbb{C}_p \otimes_{\mathbb{Z}} X_{E,S(E)})^{-1} \\ &\xrightarrow{\sim} \mathrm{d}^\diamond_{\mathbb{C}_p[\mathcal{G}_E]}(\mathbb{C}_p \otimes_{\mathbb{Z}} X_{E,S(E)}) \otimes \mathrm{d}^\diamond_{\mathbb{C}_p[\mathcal{G}_E]}(\mathbb{C}_p \otimes_{\mathbb{Z}} X_{E,S(E)})^{-1} \\ &\xrightarrow{\sim} (\mathbb{C}_p[\mathcal{G}_E], 0). \end{aligned}$$

Here the first morphism is induced by the descriptions in Lemma 5.1(ii) and (iii) and the natural passage-to-cohomology map (from [13, Prop. 5.17(i)]), the final morphism is the canonical evaluation map and the second morphism is induced by the scalar extension of the Dirichlet regulator isomorphism

$$\mathrm{Reg}_{E,S(E)} : \mathbb{R} \otimes_{\mathbb{Z}} \mathcal{O}^\times_{E,S(E)} \cong \mathbb{R} \otimes_{\mathbb{Z}} X_{E,S(E)} \tag{5.1.5}$$

that sends $u$ in $\mathcal{O}^\times_{E,S(E)}$ to $-\sum_w \log(|u|_w)\cdot w$, where in the sum $w$ runs over $S(E)_E$ and $|\cdot|_w$ denotes the absolute value with respect to $w$ (normalized as in [47, Chap. 0, 0.2]).

We write $\eta_E$ for the pre-image under $\vartheta_E$ of the element

$$\theta^*_{E,S(E)}(0) := (\textstyle\sum_{\chi\in \mathrm{Ir}(\mathcal{G}_E)} L^*_{S(E)}(\chi^{-1}, 0)e_\chi, 0)$$

of $(\mathbb{C}_p[\mathcal{G}_E]^\times, 0)$, where $L^*_{S(E)}(\chi^{-1}, 0)$ denotes the leading term in the Taylor expansion at $z = 0$ of the series $L_{S(E)}(\chi^{-1}, z)$.

Then, since $p$ is odd, Lemma 5.4 below implies that the collection

$$\eta^{\mathrm{ab}} := (\eta_E^2)_{E\in\Omega^{\mathrm{ab}}}$$

constitutes a $\Lambda^{\mathrm{ab}}$-basis of $\mathrm{VS}(\mathbb{Z}_p(1), \mathbb{Q}^{\mathrm{ab},+})$. The claimed result will therefore follow if we can show that for every $L$ in $\Omega^{\mathrm{ab}}$ one has $\Theta_L(\eta^{\mathrm{ab}}) = (\epsilon_L)^{x_L}$ with $x_L = 1$ if $p$ ramifies in $L$ and $x_L = 1 - \sigma_p$ otherwise.

To do this we fix $L$, abbreviate its conductor $f(L)$ to $f$ and write $\mathbb{Q}(f)$ for the maximal real subfield of $\mathbb{Q}(\zeta_f)$. We also write $S(f)$ for the subset of $S(L)$ comprising $\infty$ and all prime divisors of $f$ (so that $S(L) = S(f)$ if $p$ is ramified in $F$ and otherwise $S(L) \setminus S(f) = \{p\}$). We set $\mathcal{G}_f := \mathcal{G}_{\mathbb{Q}(f)}$, $x_f := x_{\mathbb{Q}(f)}$ and $e_f := e_{\mathbb{Q}(f)}$.

Then it is enough to prove the above equality with $L$ replaced by $\mathbb{Q}(f)$. The key point now is to recall (from, for example [47, Chap. 3, §5]) that for each $\chi$ in $\mathrm{Ir}(\mathcal{G}_f)$ the first derivative $L^1_{S(f)}(\chi, z)$ of $L_{S(f)}(\chi, z)$ is holomorphic at $z = 0$ and that the normalization (5.1.4) implies

$$L^1_{S(f)}(\chi, 0) = -\frac{1}{2}\sum\nolimits_{g\in\mathcal{G}_f}\chi(g)\log|\sigma(1-\zeta_f^g)^{1+\tau}|, \tag{5.1.6}$$

where $\tau$ denotes complex conjugation.

Now $L^1_{S(\mathbb{Q}(f))}(\chi, 0) = \chi(x_f)\cdot L^1_{S(f)}(\chi, 0)$ and so Lemma 5.1(v) implies that $e_\chi \cdot e_f \neq 0$ if and only if both $\chi(x_f) \neq 0$ and $L^1_{S(f)}(\chi, 0) \neq 0$. The above displayed equality therefore implies, firstly, that the image in $\mathbb{Q}\cdot\mathcal{O}^\times_{\mathbb{Q}(f),S(f)}$ of $(1-\zeta_f)^{x_f(1+\tau)/2}$ is stable under the action of the idempotent $e_f$ and then, secondly, that its image under the isomorphism (5.1.5) is equal to $(e_f \cdot \theta^*_{\mathbb{Q}(f),S(\mathbb{Q}(f))}(0)\cdot(w_{\mathbb{Q}(f)} - w_0), 0)$ where $w_0$ is any choice of $p$-adic place of $\mathbb{Q}(f)$. This latter fact then combines with the explicit definition (via (4.2.2)) of the map $\Theta_{\mathbb{Q}(f)}$ to imply the required equality

$$\Theta_{\mathbb{Q}(f)}(\eta^{\mathrm{ab}}) = e_f((1-\zeta_f)^{x_f(1+\tau)}) = (1-\zeta_f)^{(1+\tau)x_f} = (\epsilon_{\mathbb{Q}(f)})^{x_f}.$$

□

**Lemma 5.4.** *The* $\Lambda^{\mathrm{ab}}$*-module* $\mathrm{VS}(\mathbb{Z}_p(1), \mathbb{Q}^{\mathrm{ab},+})$ *is free of rank one, with basis* $(\eta_E)_{E\in\Omega^{\mathrm{ab}}}$.

*Proof.* At the outset, we fix $E$ in $\Omega^{\mathrm{ab}}$ and recall (from [8, Prop. 3.4]) that the equivariant Tamagawa Number Conjecture for the pair $(h^0(\mathrm{Spec}(E)), \mathbb{Z}_p[\mathcal{G}_E])$ asserts that $\mathrm{d}_{\mathbb{Z}_p[\mathcal{G}_E]}(C_E)$ is a free (graded) $\mathbb{Z}_p[\mathcal{G}_E]$-module with basis $\eta_E$. We further recall that, since $p$ is odd, this conjecture is known to be valid by the main result of [7].

Given the explicit definition (in Definition 4.4) of $\mathrm{VS}(\mathbb{Z}_p(1), \mathbb{Q}^{\mathrm{ab},+})$ as an inverse limit, the claimed result will therefore follow if we can show that for each pair of fields $E$ and $E'$ in $\Omega^{\mathrm{ab}}$ with $E \subseteq E'$ one has $\nu_{E'/E}(\eta_{E'}) = \eta_E$.

To prove this we note that for each place $v$ in $S(E') \setminus S(E)$ the discussion before (4.1.2) identifies $\mathsf{R}\Gamma(\kappa(v), \mathbb{Z}_p[\mathcal{G}_E])^*[-1]$ with the complex $\Psi_v$ that is equal to $\mathbb{Z}_p[\mathcal{G}_E]$ in degrees zero and one (upon which each element $g$ of $\mathcal{G}_E$ acts as multiplication by $g^{-1}$) and has the differential $x \mapsto (1-\sigma_v)x$.

We write $Y_v$ for the free abelian group on the set of places of $E$ above $v$ and, fixing a place $w_v$ of $E$ above $v$, note there are isomorphisms $\iota^i_v : H^i(\Psi_v) \cong Y_v$ for $i \in \{0, 1\}$ with $\iota^0_v(x) = |\mathcal{G}_{E,v}|^{-1}x(w_v)$ and $\iota^1_v(x) = x(w_v)$ where $\mathcal{G}_{E,v}$ denotes the decomposition subgroup of $v$ in $\mathcal{G}_E$.

The key fact now is that the $\mathrm{Gal}(E'/E)$-invariants of $\vartheta_{E'}$ differs from the composite $\vartheta_E \circ (\mathbb{C}_p \otimes_{\mathbb{Z}_p} \nu_{E'/E})$ only in that for each $v$ in $S(E') \setminus S(E)$ and $\chi$ in $\mathrm{Hom}(\mathcal{G}_E, \mathbb{C}_p^\times)$ these maps respectively use the upper and lower composite homomorphisms in the diagram

$$\begin{array}{ccccc} e_\chi(\mathbb{C}_p\cdot \mathrm{d}_{\mathbb{Z}_p[\mathcal{G}_E]}(\Psi_v)) & \xrightarrow{\alpha_1} & \mathrm{d}^\diamond_{\mathbb{C}_p}(e_\chi(\mathbb{C}_p\cdot Y_v))\otimes \mathrm{d}^\diamond_{\mathbb{C}_p}(e_\chi(\mathbb{C}_p\cdot Y_v))^{-1} & \xrightarrow{\alpha_2} & (\mathbb{C}_p[\mathcal{G}_E]e_\chi,0) \\ \| & & & & \downarrow{\cdot\epsilon_v^\chi} \\ e_\chi(\mathbb{C}_p\cdot \mathrm{d}_{\mathbb{Z}_p[\mathcal{G}_E]}(\Psi_v)) & \xrightarrow{\alpha_3} & \mathrm{d}^\diamond_{\mathbb{C}_p}(\mathbb{C}_p[\mathcal{G}_E]e_\chi)\otimes \mathrm{d}^\diamond_{\mathbb{C}_p}(\mathbb{C}_p[\mathcal{G}_E]e_\chi)^{-1} & \xrightarrow{\alpha_4} & (\mathbb{C}_p[\mathcal{G}_E]e_\chi,0). \end{array}$$

Here we abbreviate $\mathrm{d}^\diamond_{\mathbb{C}_p[\mathcal{G}_E]e_\chi}(-)$ to $\mathrm{d}^\diamond_{\mathbb{C}_p}(-)$, $\alpha_1$ is the standard 'passage to cohomology' isomorphism induced by ([13, Prop. 5.17(i)] and) the maps $\iota_v^0$ and $\iota_v^1$, $\alpha_2$ is the morphism induced by multiplication by $\log(\mathrm{N}(w_v))$ on $e_\chi(\mathbb{C}_p{\cdot}Y_v)$, $\alpha_3$ is the identification resulting from [13, (5.3.1)] and the fact that each non-zero term of $e_\chi(\mathbb{C}_p\otimes\Psi_v)$ identifies with $\mathbb{C}_p[\mathcal{G}_E]e_\chi$, $\alpha_4$ is the standard isomorphism and we have set

$$\epsilon_v^\chi := \begin{cases} 1-\chi^{-1}(\sigma_v), & \text{if } \chi(\sigma_v)\neq 1 \\ |\mathcal{G}_{E,v}|^{-1}\cdot\log(\mathrm{N}(w_v)) = \log(\mathrm{N}(v)), & \text{otherwise.}\end{cases}$$

The claimed result then follows from the fact that the argument of Lemma 4.12 implies that the above diagram commutes, whilst an explicit computation shows that for every $\chi$ in $\mathrm{Hom}(\mathcal{G}_E,\mathbb{C}_p^\times)$ one has

$$L^*_{S(E')}(\chi^{-1},0) = \left(\prod\nolimits_{v\in S(E')\setminus S(E)}\epsilon_v^\chi\right)\cdot L^*_{S(E)}(\chi^{-1},0).$$

□

5.1.3. Following Proposition 4.6 we fix a $\Lambda$-basis element $\eta'$ of $\mathrm{VS}(\mathbb{Z}_p(1),\mathbb{Q}^{c,+})$.

Then the image of $\eta'$ under the natural projection map

(5.1.7) $$\mathrm{VS}(\mathbb{Z}_p(1),\mathbb{Q}^{c,+})\to \mathrm{VS}(\mathbb{Z}_p(1),\mathbb{Q}^{\mathrm{ab},+})$$

is a basis of the latter module over $\xi(\Lambda^{\mathrm{ab}})=\Lambda^{\mathrm{ab}}$ and hence, following Theorem 5.3, equal to $v\cdot\eta^{\mathrm{ab}}$ for some element $v$ of $\Lambda^{\mathrm{ab},\times}$.

Since the natural projection map $\xi(\Lambda)^\times\to\xi(\Lambda^{\mathrm{ab}})^\times=\Lambda^{\mathrm{ab},\times}$ is surjective (by Lemma 5.5 below) we can then choose a pre-image $u$ of $v^{-1}$ under this map. The element

$$\eta := u\cdot\eta'$$

is then a pre-image of $\eta^{\mathrm{ab}}$ under the map (5.1.7).

By applying Theorem 2.19 in this setting we therefore obtain an element

(5.1.8) $$\varepsilon^{\mathrm{cyc}} := \Theta(\eta)$$

of $\mathrm{ES}_1(\mathbb{Z}_p(1),\mathbb{Q}^{c,+})$ and, with this definition, Theorem 5.3 implies directly that $\varepsilon^{\mathrm{cyc}}$ has the property stated in Theorem B(i).

**Lemma 5.5.** *The natural projection map $\Lambda\to\Lambda^{\mathrm{ab}}$ induces a surjective homomorphism from $\xi(\Lambda)^\times$ to $\xi(\Lambda^{\mathrm{ab}})^\times=\Lambda^{\mathrm{ab},\times}$.*

*Proof.* We use the same notation as in the proof of Proposition 4.6 (with $K=\mathbb{Q}$). For each natural number $n$ we also set $\Gamma_n := \mathrm{Gal}(\mathbb{Q}_{(n)}/\mathbb{Q})$.

Then, since for each $n$ the $\mathbb{Z}_p$-algebra $\xi(\mathbb{Z}_p[\Gamma_n])$ is semi-local, Bass's Theorem implies that the natural (surjective) projection map $\pi_n : \xi(\mathbb{Z}_p[\Gamma_n]) \to \xi(\mathbb{Z}_p[\Gamma_n^{\mathrm{ab}}]) = \mathbb{Z}_p[\Gamma_n^{\mathrm{ab}}]$ restricts to give a surjective homomorphism

$$\pi_n^\times : \xi(\mathbb{Z}_p[\Gamma_n])^\times \to \mathbb{Z}_p[\Gamma_n^{\mathrm{ab}}]^\times.$$

It suffices for us to show that the inverse limit over $n$ of these maps $\pi_n^\times$ is itself surjective, and for this it is enough to show, by the Mittag-Leffler criterion, that the natural projection map $\varrho_n : \xi(\mathbb{Z}_p[\Gamma_n]) \to \xi(\mathbb{Z}_p[\Gamma_{n-1}])$ is such that $\varrho_n(\ker(\pi_n^\times)) = \ker(\pi_{n-1}^\times)$.

As a first step we claim that $\varrho_n(\ker(\pi_n)) = \ker(\pi_{n-1})$. To show this we consider the exact commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker(\pi_n) & \longrightarrow & \xi(\mathbb{Z}_p[\Gamma_n]) & \xrightarrow{\pi_n} & \mathbb{Z}_p[\Gamma_n^{\mathrm{ab}}] & \longrightarrow & 0 \\
 & & \big\downarrow & & \varrho_n\big\downarrow & & \varrho_n^{\mathrm{ab}}\big\downarrow & & \\
0 & \longrightarrow & \ker(\pi_{n-1}) & \longrightarrow & \xi(\mathbb{Z}_p[\Gamma_{n-1}]) & \xrightarrow{\pi_{n-1}} & \mathbb{Z}_p[\Gamma_{n-1}^{\mathrm{ab}}] & \longrightarrow & 0
\end{array}$$

in which the first vertical arrow is the restriction of $\varrho_n$ and $\varrho_n^{\mathrm{ab}}$ is the natural projection map. In particular, since $\varrho_n$ is surjective, the Snake Lemma implies it is enough to show that $\pi_n(\ker(\varrho_n)) = \ker(\varrho_n^{\mathrm{ab}})$.

Next we note that the kernel of $\varrho_n^{\mathrm{ab}}$ is generated over $\mathbb{Z}_p$ by elements of the form $\gamma(\delta-1)$ where $\gamma$ belongs to $\Gamma_n^{\mathrm{ab}}$ and $\delta$ to the kernel of the projection $\Gamma_n^{\mathrm{ab}} \to \Gamma_{n-1}^{\mathrm{ab}}$.

We choose elements $\gamma'$ and $\delta'$ of $\Gamma_n$ which project to $\gamma$ and $\delta$ in $\Gamma_n^{\mathrm{ab}}$. Then $\varrho_n(\delta')$ belongs to the commutator subgroup of $\Gamma_{n-1}$ and so we can choose a finite set of commutators $\{[\delta_{i1}', \delta_{i2}']\}_{1\le i\le m}$ in $\Gamma_n$ such that the element $\delta'\prod_{1\le i\le m}[\delta_{i1}',\delta_{i2}']^{-1}$ projects to the identity element of $\Gamma_{n-1}$. It is then easy to check that, writing $M_1$ and $M_2$ for the $1\times 1$ matrices with entries $\gamma'\delta'\prod_{1\le i\le m}[\delta_{i1}',\delta_{i2}']^{-1}$ and $\gamma'$ respectively, the element $\mathrm{Nrd}_{\mathbb{Q}[\Gamma_n]}(M_1) - \mathrm{Nrd}_{\mathbb{Q}[\Gamma_n]}(M_2)$ of $\xi(\mathbb{Z}_p[\Gamma_n])$ belongs to $\ker(\varrho_n)$ and is sent by $\pi_n$ to $\gamma(\delta-1)$. It follows that $\pi_n(\ker(\varrho_n)) = \ker(\varrho_n^{\mathrm{ab}})$, as claimed above.

To proceed we now decompose $\xi(\mathbb{Z}_p[\Gamma_n])$ as a finite product of local rings $\prod_{j\in J} R_j$. Then for each index $j$ there are ideals $I_j, I_j'$ and $I_j^\dagger$ of $R_j$ such that $\ker(\pi_n) = \prod_{j\in J} I_j^\dagger$, $\xi(\mathbb{Z}_p[\Gamma_{n-1}]) = \prod_{j\in J} R_j/I_j$ and $\ker(\pi_{n-1}) = \prod_{j\in J} I_j'/I_j$.

In addition, for any index $j$ and any proper ideal $I$ of $R_j$ one has $1 + I \subset R_j^\times$ and so

$$\ker(\pi_n^\times) = \prod\nolimits_{j\in J_1} R_j^\times \times \prod\nolimits_{j\in J\setminus J_1} (1 + I_j^\dagger)$$

and

$$\ker(\pi_{n-1}^\times) = \prod\nolimits_{j\in J_2} (R_j/I_j)^\times \times \prod\nolimits_{j\in J_3\setminus J_2} (1 + I_j'/I_j),$$

where we set

$$J_i := \begin{cases} \{j\in J : I_j^\dagger = R_j\}, & \text{if } i = 1,\\ \{j \in J : I_j' = R_j \neq I_j\}, & \text{if } i = 2,\\ \{j\in J : I_j' \neq I_j\}, & \text{if } i = 3.\end{cases}$$

Now, since $\varrho_n(\ker(\pi_n)) = \ker(\pi_{n-1})$, one has $J_2 \subseteq J_1$, $J_3\setminus J_2 \subseteq J \setminus J_1$ and $I_j^\dagger + I_j = I_j'$ for all $j \in J_3 \setminus J_2$. These facts then combine with the above decompositions to imply that $\varrho_n(\ker(\pi_n^\times)) = \ker(\pi_{n-1}^\times)$, as required to complete the proof. $\square$

5.2. **The proof of Theorem B.** It remains to check that the system $\varepsilon^{\rm cyc}$ defined by (5.1.8) has the properties described in Theorem B(ii) and (iii).

Regarding the field $F$ as fixed we henceforth set $G := \mathcal{G}_F$ and $S := S(F)$. In the sequel we also write $M^*$ for the linear dual $\mathrm{Hom}_{\mathbb{Z}_p}(M, \mathbb{Z}_p)$ of a $\mathbb{Z}_p[G]$-module $M$ and regard it as endowed with the contragredient action of $\mathbb{Z}_p[G]$.

5.2.1. The fact that $\varepsilon^{\rm cyc}$ has the property in Theorem B(ii) follows directly from the following stronger result. In this result we use the notion of transpose (non-commutative) Fitting invariant from [13, §3.4.2]. For each $\varphi$ in $\mathrm{Hom}_{\mathbb{Q}_p[G]}(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathcal{O}^\times_{F,S,p}, \mathbb{Q}_p[G])$ we also set

$$\bigwedge\nolimits^1_{\mathbb{Q}_p[G]} \varphi := \wedge_{i=1}^{i=1} \varphi_i,$$

where $\varphi_1 := \varphi$ and the expression on the right hand side is defined via [13, (4.2.7) and (4.2.3)].

**Proposition 5.6.** *For each $\varphi$ in $\mathrm{Hom}_{\mathbb{Z}_p[G]}(\mathcal{O}^\times_{F,S,p}, \mathbb{Z}_p[G])$ and each prime $\ell$ in $S$ one has*

$$\big(\bigwedge\nolimits^1_{\mathbb{Q}_p[G]} \varphi\big)(\varepsilon^{\rm cyc}_F) \in \mathrm{Fit}^{\mathrm{tr},1}_{\mathbb{Z}_p[G]}(\mathrm{Sel}_S(F)_p)^{\#} \cap \mathrm{pAnn}_{\mathbb{Z}_p[G]}(\mathrm{Cl}(\mathcal{O}_F[1/\ell])_p).$$

*Proof.* Set $\epsilon := \varepsilon^{\rm cyc}_F$. Then for each $\varphi$ in $\mathrm{Hom}_{\mathbb{Z}_p[G]}(\mathcal{O}^\times_{F,S,p}, \mathbb{Z}_p[G])$ the equality (4.3.2) that is used to prove Theorem 2.19(ii) implies the existence of an element $u$ of $\xi(\mathbb{Z}_p[G])^\times$ that is independent of $\varphi$ and such that

$$\big(\bigwedge\nolimits^1_{\mathbb{Q}_p[G]} \varphi\big)(\epsilon) = u \cdot \mathrm{Nrd}_{\mathbb{Q}_p[G]}\big((M_\varphi \mid M_F)\big). \tag{5.2.1}$$

Here $M_\varphi$ is the $d \times 1$ column vector with $i$-th entry equal to $\varphi(b_{i,F})$ and $M_F$ is the matrix in $\mathrm{M}_{d,d-1}(\mathbb{Z}_p[G])$ such that the block matrix $\big(0_{d,1} \mid M_F\big)$ is the matrix of the differential in the complex $P^\bullet_F$ in Lemma 4.10(iii) with respect to the basis $\{b_{i,F}\}_{1\le i\le d}$ of $P_F$.

Now the sequence (4.2.5) gives a free presentation $\Pi$ of the $\mathbb{Z}_p[G]$-module $H^1(C_{F,S}(\mathbb{Z}_p(1)))$ and Lemma 5.1(iv) implies that the transpose $\Pi^{\rm tr}$ of $\Pi$ (in the sense of [13, Def. 3.21]) is a presentation of the $\mathbb{Z}_p[G]$-module $\mathrm{Sel}_S(F)_p$. One therefore has

$$\big(\bigwedge\nolimits^1_{\mathbb{Q}_p[G]} \varphi\big)(\epsilon) \in \mathrm{Fit}^1_{\mathbb{Z}_p[G]}(\Pi) = \mathrm{Fit}^{\mathrm{tr},1}_{\mathbb{Z}_p[G]}(\Pi^{\rm tr})^{\#} \subseteq \mathrm{Fit}^{\mathrm{tr},1}_{\mathbb{Z}_p[G]}(\mathrm{Sel}_S(F)_p)^{\#}$$

where the containment follows directly from (4.3.2) and the equality from [13, Lem. 3.23].

To prove the second containment in the claimed result we first note that (5.2.1) implies

$$\begin{aligned}\big(\bigwedge\nolimits^1_{\mathbb{Q}_p[G]} \varphi\big)(\epsilon)^{\#} &= u^{\#} \cdot \mathrm{Nrd}_{\mathbb{Q}_p[G]}\big((M_\varphi \mid M_F)\big)^{\#} \\ &= u^{\#} \cdot \mathrm{Nrd}_{\mathbb{Q}_p[G]}\Big(\begin{pmatrix} \iota_{\#}(M_\varphi^{\rm tr}) \\ \iota_{\#}(M_F^{\rm tr}) \end{pmatrix}\Big).\end{aligned} \tag{5.2.2}$$

Here, for each matrix $N$ in $\mathrm{M}_{b,b'}(\mathbb{Z}_p[\Gamma])$ we write $\iota_{\#}(N)$ for the matrix obtained by applying to each component of $N$ the $\mathbb{Z}_p$-linear anti-involution of $\mathbb{Z}_p[\Gamma]$ that inverts elements of $\Gamma$. In particular, the second equality in the above display follows directly from [13, (3.4.1)].

Now the argument used above implies that the block matrix

$$\begin{pmatrix} 0_{1,d} \\ \hline \iota_{\#}(M_F^{\mathrm{tr}}) \end{pmatrix}$$

represents, with respect to the dual basis $\{b_{i,F}^*\}_{1\le i\le d}$ of $P_F^*$, the endomorphism $\theta_F^*$ in an exact sequence of the form

$$P_F^* \xrightarrow{\theta_F^*} P_F^* \xrightarrow{\pi} \mathrm{Sel}_S(F)_p \to 0.$$

It is also easily checked that $\pi$ sends the element of $P_F^*$ whose co-ordinate vector with respect to $\{b_{i,F}^*\}_{1\le i\le d}$ is $\iota_{\#}(M_\varphi)$ to an element $\hat{\varphi}$ that the homomorphism $\mathrm{Sel}_S(F)_p \to (\mathcal{O}_{F,S,p}^\times)^*$ in (5.1.1) sends to $\varphi$. The block matrix

$$\begin{pmatrix} \iota_{\#}(M_\varphi^{\mathrm{tr}}) \\ \hline \iota_{\#}(M_F^{\mathrm{tr}}) \end{pmatrix}$$

is therefore the matrix of a free presentation of the quotient module $\mathrm{Sel}_S(F)_p/(\mathbb{Z}_p[G]\cdot\hat{\varphi})$ and so (5.2.2) combines with [13, Th. 3.20(iii)] to imply that

$$\big({\textstyle\bigwedge}^1_{\mathbb{Q}_p[G]}\varphi\big)(\epsilon)^{\#} \in \mathrm{pAnn}_{\mathbb{Z}_p[G]}\big(\mathrm{Sel}_S(F)_p/(\mathbb{Z}_p[G]\cdot\hat{\varphi})\big). \tag{5.2.3}$$

We now write $\Sigma$ for the subset $\{\infty,\ell\}$ of $S$ and recall that the results of [8, Prop. 2.4(ii) and (iii)] combine to imply the existence of a canonical surjective homomorphism of $\mathbb{Z}_p[G]$-modules $\rho_\ell' : \mathrm{Sel}_S(F)_p \to \mathrm{Sel}_\Sigma(F)_p$ and hence also a surjective homomorphism

$$\mathrm{Sel}_S(F)_p/(\mathbb{Z}_p[G]\cdot\hat{\varphi}) \twoheadrightarrow \mathrm{Sel}_\Sigma(F)_p/(\mathbb{Z}_p[G]\cdot\rho_\ell'(\hat{\varphi})). \tag{5.2.4}$$

In addition, since the exact sequences (5.1.1) imply that $\mathbb{Q}_p\otimes_{\mathbb{Z}_p}\rho_\ell'$ is equal to the scalar extension of the natural restriction map $\rho_\ell : (\mathcal{O}_{F,S,p}^\times)^* \to (\mathcal{O}_{F,\Sigma,p}^\times)^*$, the result of Lemma 5.7 below allows us to assume that $\rho_\ell'(\hat{\varphi})$ spans a free $\mathbb{Q}_p[G]$-module.

In particular, in this case $\mathbb{Z}_p[G]\cdot\rho_\ell'(\hat{\varphi})$ is torsion-free and so the natural map

$$\mathrm{Sel}_\Sigma(F)_{p,\mathrm{tor}} \to \mathrm{Sel}_\Sigma(F)_p/(\mathbb{Z}_p[G]\cdot\rho_\ell'(\hat{\varphi}))$$

from the $\mathbb{Z}$-torsion subgroup of $\mathrm{Sel}_\Sigma(F)_p$ is injective. Since $\mathrm{Sel}_\Sigma(F)_{p,\mathrm{tor}}$ is isomorphic to $\mathrm{Cl}(\mathcal{O}_F[1/\ell])_p^\vee$ (by (5.1.1)), the homomorphism (5.2.4) therefore implies that the quotient module $\mathrm{Sel}_S(F)_p/(\mathbb{Z}_p[G]\cdot\hat{\varphi})$ has a subquotient isomorphic to $\mathrm{Cl}(\mathcal{O}_F[1/\ell])_p^\vee$ and so the containment (5.2.3) implies that

$$\big({\textstyle\bigwedge}^1_{\mathbb{Q}_p[G]}\varphi\big)(\epsilon)^{\#} \in \mathrm{pAnn}_{\mathbb{Z}_p[G]}\big(\mathrm{Cl}(\mathcal{O}_F[1/\ell])_p^\vee\big) = \mathrm{pAnn}_{\mathbb{Z}_p[G]}\big(\mathrm{Cl}(\mathcal{O}_F[1/\ell])_p\big)^{\#},$$

where the last equality follows from [13, Rem. 3.24]. Upon applying the involution $x\mapsto x^{\#}$ to this containment we deduce the second containment in the statement of this result, as required to complete the proof. □

In the next two results we set $U := \mathcal{O}_{F,S,p}^\times$.

**Lemma 5.7.** *It suffices to prove the second containment of Proposition 5.6 in the case that $\rho_\ell(\varphi)$ spans a free $\mathbb{Q}_p[G]$-module.*

*Proof.* We set $\epsilon := \varepsilon_F^{\rm cyc}$ and claim first that there exists a natural number $n$ with the property that if $\varphi$ and $\varphi'$ are any elements of $U^*$, then one has

$$\varphi - \varphi' \in n\cdot U^* \Longrightarrow \big(\bigwedge\nolimits^1_{\mathbb{Q}_p[G]}\varphi\big)(\epsilon) - \big(\bigwedge\nolimits^1_{\mathbb{Q}_p[G]}\varphi'\big)(\epsilon) \in |{\rm Cl}(\mathcal{O}_F[1/\ell])|\cdot \xi(\mathbb{Z}_p[G]). \tag{5.2.5}$$

To see this, we note that the equality (5.2.1) implies that

$$\big(\bigwedge\nolimits^1_{\mathbb{Q}_p[G]}\varphi\big)(\epsilon) - \big(\bigwedge\nolimits^1_{\mathbb{Q}_p[G]}\varphi'\big)(\epsilon) = u\cdot\big({\rm Nrd}_{\mathbb{Q}_p[G]}\big((M_\varphi \mid M_L)\big) - {\rm Nrd}_{\mathbb{Q}_p[G]}\big((M_{\varphi'} \mid M_L)\big)\big)$$

where $M_{\varphi'}$ is the $d\times 1$ column vector with $i$-th entry $\varphi'(b_{i,F})$. To prove (5.2.5) it is therefore enough to note there exists a natural number $n$ such that for any matrices $M$ and $M'$ in ${\rm M}_d(\mathbb{Z}_p[G])$ one has

$$M - M' \in n\cdot {\rm M}_d(\mathbb{Z}_p[G]) \Longrightarrow {\rm Nrd}_{\mathbb{Q}_p[G]}(M) - {\rm Nrd}_{\mathbb{Q}_p[G]}(M') \in |{\rm Cl}(\mathcal{O}_F[1/\ell])|\cdot \mathbb{Z}_p[G].$$

(cf. the proof of [13, Lem. 3.2(ii)]).

With $n$ as in (5.2.5), we can then apply the result of Lemma 5.8 below to deduce the existence for any given $\varphi$ in $U^*$ of an element $\varphi'$ of $U^*$ with the property that $\rho_\ell(\varphi')$ spans a free $\mathbb{Q}_p[G]$-module and, in addition, one has

$$\big(\bigwedge\nolimits^1_{\mathbb{Q}_p[G]}\varphi\big)(\epsilon) - \big(\bigwedge\nolimits^1_{\mathbb{Q}_p[G]}\varphi'\big)(\epsilon) \in |{\rm Cl}(\mathcal{O}_F[1/\ell])|\cdot \xi(\mathbb{Z}_p[G]).$$

Since any element of $|{\rm Cl}(\mathcal{O}_F[1/\ell])|\cdot \xi(\mathbb{Z}_p[G])$ belongs to ${\rm pAnn}_{\mathbb{Z}_p[G]}({\rm Cl}(\mathcal{O}_F[1/\ell])_p)$, the claimed result is therefore clear. □

**Lemma 5.8.** *Fix $\varphi$ in $U^*$ and a natural number $n$. Then there exists $\varphi'$ in $U^*$ such that $\varphi' - \varphi \in n\cdot U^*$ and $\rho_\ell(\varphi')$ spans a free $\mathbb{Q}_p[G]$-module.*

*Proof.* Set $V := \mathcal{O}^\times_{F,\Sigma,p}$. Then, since $F$ is totally real, we may choose a free $\mathbb{Z}_p[G]$-submodule $\mathcal{F}$ of $V^*$ of rank one. We then choose a homomorphism $f$ in $U^*$ with $\mathbb{Q}_p[G]\cdot\rho_\ell(f) = \mathbb{Q}_p\cdot\mathcal{F}$.

For any integer $m$ we set $\varphi_m := \varphi + mnf$ and note it suffices to show that for any sufficiently large $m$ the element $\rho_\ell(\varphi_m)$ spans a free $\mathbb{Q}_p[G]$-module.

Consider the composite homomorphism of $\mathbb{Q}_p[G]$-modules $\mathbb{Q}_p\cdot\mathcal{F} \to \mathbb{Q}_p\cdot V^* \to \mathbb{Q}_p\cdot\mathcal{F}$ where the first arrow sends the basis element $\rho_\ell(f)$ to $\rho_\ell(\varphi_m)$ and the second is induced by a choice of $\mathbb{Q}_p[G]$-equivariant section to the projection $\mathbb{Q}_p\cdot V^* \to \mathbb{Q}_p\cdot\big(V^*/\mathcal{F}\big)$.

This map sends $\rho_\ell(f)$ to $(\lambda_\varphi + mn)\cdot\rho_\ell(f)$ for an element $\lambda_\varphi$ of $\mathbb{Q}_p[G]$ that is independent of $m$. In particular, if $m$ is large enough to ensure $\lambda_\varphi - mn$ is invertible in $\mathbb{Q}_p[G]$, then the composite homomorphism is injective and so $\rho_\ell(\varphi_m)$ must span a free $\mathbb{Q}_p[G]$-module, as required. □

5.2.2. In order to prove Theorem B(iii) we must first specify the map ${\rm Reg}_F^\chi$ that occurs in that result.

The representation $\rho_\chi$ fixed in §2.1.1 gives rise to a (left) $\mathbb{C}_p[G]$-module $V_\chi$ of character $\chi$ and we write $V_\chi^*$ for ${\rm Hom}_{\mathbb{C}_p}(V_\chi, \mathbb{C}_p)$, endowed with the natural right action of $\mathbb{C}_p[G]$. For each $\mathbb{Z}_p[G]$-module $M$ we write $\mathbb{C}_p\cdot M$ for the $\mathbb{C}_p[G]$-module generated by $M$ and consider the associated $\mathbb{C}_p$-vector space

$$(\mathbb{C}_p\cdot M)^\chi := V_\chi^* \otimes_{\mathbb{C}_p[G]} (\mathbb{C}_p\cdot M).$$

In particular, setting $U := \mathcal{O}^\times_{F,S,p}$ and $X := X_{F,S,p}$ we write

$$\mathrm{Reg}^\chi_F : (\mathbb{C}_p \cdot U)^\chi \cong (\mathbb{C}_p \cdot X)^\chi$$

for the isomorphism of $\mathbb{C}_p$-vector spaces that is induced by the Dirichlet regulator map $\mathrm{Reg}_F = \mathrm{Reg}_{F,S}$ recalled in (5.1.5).

Now, in view of the description of $H^0(C_F)$ given in Lemma 5.1(ii), the construction of Lemma 4.10 (with $T = \mathbb{Z}_p(1)$ and $L = F$) gives an exact sequence of $\mathbb{Z}_p[G]$-modules

$$0 \to U \xrightarrow{\iota} P \xrightarrow{\theta} P \xrightarrow{\pi} H^1(C_F) \to 0. \tag{5.2.6}$$

Since $Y_F(\mathbb{Z}_p) = Y_{F,S_\infty(F),p}$ we can also ensure that the $\mathbb{Z}_p[G]$-basis $\{b_i\}_{1\le i\le d}$ of $P$ fixed in Lemma 4.10(ii) is such that the surjective composite homomorphism

$$\varpi : P \xrightarrow{\pi} H^1(C_F) \to X,$$

in which the second map comes from the exact sequence in Lemma 5.1(iii), sends $b_1$ to $w_{\infty,F} - w_{p,F}$.

Since the algebra $\mathbb{Q}_p[G]$ is semisimple we can fix $\mathbb{Q}_p[G]$-equivariant sections $\iota_1$ and $\iota_2$ to the surjections $\mathbb{Q}_p \cdot P \to \mathbb{Q}_p \cdot \mathrm{im}(\theta)$ and $\mathbb{Q}_p \cdot P \to \mathbb{Q}_p \cdot X$ that are respectively induced by $\theta$ and $\varpi$ and we can assume that $\iota_2(w_{\infty,F} - w_{p,F}) = b_1$. These sections then give a direct sum decomposition of $\mathbb{C}_p[G]$-modules

$$\mathbb{C}_p \cdot P = (\mathbb{C}_p \cdot U) \oplus (\mathbb{C}_p \otimes_{\mathbb{Q}_p} \iota_1)(\mathbb{C}_p \cdot \mathrm{im}(\theta))$$

and, via this decomposition, we define $\langle \theta, \iota_1, \iota_2\rangle$ to be the automorphism of $\mathbb{C}_p \cdot P$ that is equal to $(\mathbb{C}_p \otimes_{\mathbb{Q}_p} \iota_2) \circ (\mathbb{C}_p \otimes_{\mathbb{R}} \mathrm{Reg}_F)$ on $\mathbb{C}_p \cdot U$ and to $\mathbb{C}_p \otimes_{\mathbb{Z}_p} \theta$ on $(\mathbb{C}_p \otimes_{\mathbb{Q}_p} \iota_1)(\mathbb{C}_p \cdot \mathrm{im}(\theta))$.

Finally we use the canonical Wedderburn decomposition $\zeta(\mathbb{C}_p[G]) = \prod_{\mathrm{Ir}_p(G)} \mathbb{C}_p$ to decompose every element $x$ of $\zeta(\mathbb{C}_p[G])$ as a vector $(x_\psi)_{\psi \in \mathrm{Ir}_p(G)}$ with each $x_\psi$ in $\mathbb{C}_p$.

The key point in the proof of Theorem B(iii) is then the following observation.

**Lemma 5.9.** *One has $e_\chi(\varepsilon^{\mathrm{cyc}}_F) \neq 0$ if and only if $L^{\chi(1)}_S(\check{\chi}, 0) \neq 0$. In addition, if $e_\chi(\varepsilon^{\mathrm{cyc}}_F)$ does not vanish, then the element $u_{F,\chi}$ of $\mathbb{C}_p$ that is uniquely specified by the displayed equality in Theorem B(iii) is non-zero and there exists a unit $v$ of $\xi(\mathbb{Z}_p[G])$ that is independent of $\chi$ and such that $u_{F,\chi} = v_\chi \cdot (\mathrm{Nrd}_{\mathbb{C}_p[G]}(\langle \theta, \iota_1, \iota_2\rangle))_\chi \cdot L^*_S(\check{\chi}, 0)^{-1}$, where $L^*_S(\check{\chi}, 0)$ denotes the leading coefficient in the Laurent expansion of $L_S(\check{\chi}, z)$ at $z = 0$.*

*Proof.* Given the explicit definition of $\varepsilon^{\mathrm{cyc}}_F$, the first claim follows directly from the proof of Theorem 4.7(i) and the description of the idempotent $e_F$ given in Lemma 5.1(v).

We therefore assume $e_\chi(\varepsilon^{\mathrm{cyc}}_F) \neq 0$ and hence that the element $u_{F,\chi}$ is both non-zero and uniquely specified by the equality in Theorem B(iii).

To compute $u_{F,\chi}$ we note that the element $y := w_{\infty,F} - w_{p,F}$ generates a free $\mathbb{Z}_p[G]$-module direct summand $Y$ of $X$.

In addition, since $L^{\chi(1)}_S(\check{\chi}, 0)$ does not vanish, the formula (5.1.2) implies both that $L^{\chi(1)}_S(\check{\chi}, 0) = L^*_S(\check{\chi}, 0)$ and $\dim_{\mathbb{C}_p}((\mathbb{C}_p \cdot X)^\chi) = \chi(1)$ and then the isomorphism $\mathrm{Reg}^\chi_F$ implies that $\dim_{\mathbb{C}_p}((\mathbb{C}_p \cdot U)^\chi) = \chi(1)$. The $\mathbb{C}_p$-spaces $\bigwedge^{\chi(1)}_{\mathbb{C}_p}(\mathbb{C}_p \cdot X)^\chi$ and $\bigwedge^{\chi(1)}_{\mathbb{C}_p}(\mathbb{C}_p \cdot U)^\chi$ are therefore of dimension one and have as respective bases the elements $e_\chi(\bigwedge^1_{\mathbb{C}_p[G]} y)$ and $e_\chi(\varepsilon^{\mathrm{cyc}}_F)$.

In particular, if we define $\lambda$ to be the non-zero element of $\mathbb{C}_p\cdot e_\chi$ that is specified by

$$(\bigwedge\nolimits_{\mathbb{C}_p}^{\chi(1)}\mathrm{Reg}_F^\chi)(e_\chi(\varepsilon_F^{\mathrm{cyc}}))=\lambda(\bigwedge\nolimits_{\mathbb{C}_p[\mathcal{G}_F]}^1 y),$$

then the displayed equality in Theorem B(iii) implies $u_{F,\chi}\cdot e_\chi=\lambda\cdot L_S^*(\check{\chi},0)^{-1}$. It is therefore enough to prove the existence of a unit $v$ of $\xi(\mathbb{Z}_p[G])$ that is independent of $\chi$ and such that

$$\lambda=e_\chi(v\cdot\mathrm{Nrd}_{\mathbb{C}_p[G]}(\langle\theta,\iota_1,\iota_2\rangle)). \tag{5.2.7}$$

To do this we use the composite homomorphism of $\mathbb{C}_p[G]$-modules

$$\Theta:\mathbb{C}_p\cdot P\to\mathbb{C}_p\cdot U\to\mathbb{C}_p\cdot X,$$

where the first and second maps are respectively induced by $\iota_1$ and $\mathrm{Reg}_F$. We also fix a section $\varrho:\mathbb{C}_p\cdot X\to\mathbb{C}_p\cdot Y$ to the inclusion $\mathbb{C}_p\cdot Y\subseteq\mathbb{C}_p\cdot X$ and use the composite homomorphism $\Theta_1:=y^*\circ\varrho\circ\Theta:\mathbb{C}_p\cdot P\to\mathbb{C}_p[G]$. Then one computes that

$$\begin{aligned}\lambda=\big(\bigwedge\nolimits_{\mathbb{C}_p[G]}^1 y^*\big)\big(\lambda(\bigwedge\nolimits_{\mathbb{C}_p[G]}^1 y)\big)&=\big(\bigwedge\nolimits_{\mathbb{C}_p[G]}^1 y^*\big)\big((\bigwedge\nolimits_{\mathbb{C}_p}^{\chi(1)}\mathrm{Reg}_F^\chi)(e_\chi(\varepsilon_F^{\mathrm{cyc}}))\big)\\&=\big(\bigwedge\nolimits_{\mathbb{C}_p[G]}^1(y^*\circ\mathrm{Reg}_F)\big)(e_\chi(\varepsilon_F^{\mathrm{cyc}}))\\&=e_\chi\big((\bigwedge\nolimits_{\mathbb{C}_p[G]}^1(y^*\circ\Theta))(\varepsilon_F^{\mathrm{cyc}})\big),\\&=e_\chi\big((\bigwedge\nolimits_{\mathbb{C}_p[G]}^1\Theta_1)(\varepsilon_F^{\mathrm{cyc}})\big).\end{aligned} \tag{5.2.8}$$

Here the first equality follows directly from [13, Lem. 4.10] and the second from the definition of $y$. In addition, the third equality follows from the explicit construction of the reduced exterior products $\bigwedge_{\mathbb{C}_p[G]}^1(-)$, the fourth from the fact $\varepsilon_F^{\mathrm{cyc}}$ belongs to $\bigwedge_{\mathbb{C}_p[G]}^1(\mathbb{C}_p\cdot U)$ and the last from the fact the surjective map $e_\chi(\mathbb{C}_p\cdot X)\to e_\chi(\mathbb{C}_p\cdot Y)$ induced by $\varrho$ is bijective since both $\dim_{\mathbb{C}_p}((\mathbb{C}_p\cdot X)^\chi)$ and $\dim_{\mathbb{C}_p}((\mathbb{C}_p\cdot Y)^\chi)=\dim_{\mathbb{C}_p}(V_\chi^*)$ are equal to $\chi(1)$.

On the other hand, the argument used to prove (5.2.1) implies the existence of a unit $v$ of $\xi(\mathbb{Z}_p[G])$ (that is independent of $\chi$ and) such that

$$\big(\bigwedge\nolimits_{\mathbb{C}_p[G]}^1\Theta_1\big)(\varepsilon_F^{\mathrm{cyc}})=v\cdot\mathrm{Nrd}_{\mathbb{C}_p[G]}\big((M'\mid M)\big).$$

Here $M'$ is the $d\times 1$ column vector with $i$-th entry equal to $\Theta_1(b_i)$. In addition, if $P'$ is the submodule of $P$ generated by $\{b_i\}_{2\le i\le d}$, then one has $\mathrm{im}(\theta)\subseteq P'$ and $M$ is the matrix in $\mathrm{M}_{d,d-1}(\mathbb{Z}_p[G])$ that represents $\theta:P\to P'$ with respect to the bases $\{b_i\}_{1\le i\le d}$ and $\{b_i\}_{2\le i\le d}$.

The key point now is that, since the section $\iota_2$ is chosen so that $\iota_2(y)=b_1$, an explicit check shows $e_\chi\cdot\big(M'\mid M\big)$ to be the matrix, with respect to the basis $\{e_\chi(b_i)\}_{1\le i\le d}$, of the endomorphism of the $\mathbb{C}_p\cdot e_\chi$-module $e_\chi(\mathbb{C}_p\cdot P)$ that is induced by $\langle\theta,\iota_1,\iota_2\rangle$.

Given these facts, the equality (5.2.7) follows directly from the computation (5.2.8). □

**Remark 5.10.** The general result of [5, Lem. A.1(iii)] combines with Lemma 4.10(iii) (with $L=F$) to imply that the equivariant Tamagawa number conjecture for $(h^0(\mathrm{Spec}(F)),\mathbb{Z}_p[G])$ is valid if and only if the element

$$\mathfrak{z}:=\big((\mathrm{Nrd}_{\mathbb{C}_p[G]}(\langle\theta,\iota_1,\iota_2\rangle))_\chi\cdot L_S^*(\check{\chi},0)^{-1}\big)_{\chi\in\mathrm{Ir}_p(G)}$$

of $\zeta(\mathbb{C}_p[G])^\times$ belongs to $\mathrm{Nrd}_{\mathbb{Q}_p[G]}(K_1(\mathbb{Z}_p[G]))$. The observations made in [5, Rem. 6.1.1] therefore imply that if $\mathcal{M}$ is any maximal $\mathbb{Z}_p$-order in $\mathbb{Q}_p[G]$ with $\mathbb{Z}_p[G] \subseteq \mathcal{M}$, then the 'Strong-Stark Conjecture' of Chinburg [16, Conj. 2.2] implies $\mathfrak{z}$ belongs to $\mathrm{Nrd}_{\mathbb{Q}_p[G]}(K_1(\mathcal{M}))$. In view of the formula in Lemma 5.9, it would therefore follow from the latter conjecture that if $e_\chi(\varepsilon_F^{\mathrm{cyc}}) \neq 0$, then the element $u_{F,\chi} = v_\chi \cdot \mathfrak{z}_\chi$ that occurs in Theorem B(iii) is a unit of the valuation ring of any finite extension of $\mathbb{Q}_p$ over which $\chi$ can be realised.

5.2.3. If $\chi(1) = 1$, then an explicit comparison of the definition of $u_{F,\chi}$ (via the displayed equality in Theorem B(iii)) with the definition of $\mathrm{Reg}_F^\chi$ and the equality (5.1.6) implies directly that $u_{F,\chi} = 2$.

To proceed with the proof of Theorem B(iii) we therefore assume in the sequel that $\chi(1) > 1$. We then note the character $\sum_{\omega\in\mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})}\chi^\omega$ takes values in $\mathbb{Q}$ and hence, by Artin's Induction Theorem (see [47, Chap. II, Th. 1.2]), there exists a natural number $m$ and an integer $n_L$ for each subfield $L$ of $F$ for which there is an equality of characters

$$m \cdot \sum_{\omega\in\mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})}\chi^\omega = \sum_{L\subseteq F} n_L \cdot 1_L^F,$$

where we write $1_L^F$ for the induction to $G$ of the trivial character $1_L$ of $G(L) := \mathrm{Gal}(F/L)$.

In particular, if we consider each of the maps from $\mathrm{Ir}_p(G)$ to $\mathbb{C}_p^\times$ that are respectively given by sending each $\psi$ to $u_{F,\psi}$, each $\psi$ to $v_\psi$, each $\psi$ to $(\mathrm{Nrd}_{\mathbb{C}_p[G]}(\langle\theta,\iota_1,\iota_2\rangle))_\psi$ and each $\psi$ to $L_S^*(\check{\psi},0)$ as homomorphisms from the group of virtual $\mathbb{C}_p$-valued characters of $G$ to $\mathbb{C}_p^\times$ (in the obvious way), then the result of Lemma 5.9 implies that

$$\left(\prod_{\omega\in\mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})} u_{F,\chi^\omega}\right)^m = \prod_{L\subseteq F}\left(\frac{v_{1_L^F}\cdot(\mathrm{Nrd}_{\mathbb{C}_p[G]}(\langle\theta,\iota_1,\iota_2\rangle))_{1_L^F}}{L_S^*(1_L^F,0)}\right)^{n_L}. \tag{5.2.9}$$

Now the behaviour of Artin $L$-series under induction of characters implies that

$$L_S^*(1_L^F,0) = \zeta_{L,S}^*(0) = -(1/2)\cdot R_{L,S}\cdot|\mathrm{Cl}(\mathcal{O}_{L,S})|, \tag{5.2.10}$$

where we write $\zeta_{L,S}^*(0)$ for the leading coefficient in the Laurent expansion at $z=0$ of the $S$-truncated zeta function of $L$ and $R_{L,S}$ for the $S$-regulator of $L$, and the second equality follows directly from the analytic class number formula for $L$ (as recalled, for example, in [47, Chap. I, Cor. 2.2]).

Next we recall that there exists a commutative diagram

$$\begin{tikzcd} K_1(\mathbb{C}_p[G]) \arrow[r] \arrow[d, "\mathrm{Nrd}_{\mathbb{C}_p[G]}"'] & K_1(\mathbb{C}_p[G(L)]) \arrow[d, "\mathrm{Nrd}_{\mathbb{C}_p[G(L)]}"] \\ \zeta(\mathbb{C}_p[G])^\times \arrow[r] & \zeta(\mathbb{C}_p[G(L)])^\times \end{tikzcd} \tag{5.2.11}$$

in which the upper and lower horizontal maps are respectively the natural restriction of scalars map and the composite homomorphism

$$\zeta(\mathbb{C}_p[G])^\times = \prod_{\mathrm{Ir}_p(G)}\mathbb{C}_p^\times \xrightarrow{\mathrm{i}_F^L} \prod_{\mathrm{Ir}_p(G(L))}\mathbb{C}_p^\times = \zeta(\mathbb{C}_p[G(L)])^\times$$

where for each $x = (x_\chi)_\chi$ in $\zeta(\mathbb{C}_p[G])^\times$ and each $\phi$ in $\mathrm{Ir}_p(G(L))$ one has

$$\mathrm{i}_L^F(x)_\phi = \prod\nolimits_{\chi \in \mathrm{Ir}_p(G)} x_\chi^{\langle \chi, \mathrm{Ind}_{G(L)}^G \phi\rangle_G}$$

with $\langle \cdot, \cdot \rangle_G$ the standard scalar product on characters of $G$.

We use the central idempotent of $\mathbb{Q}_p[G(L)]$ given by

$$e_{G(L)} := \frac{1}{|G(L)|} \sum\nolimits_{g \in G(L)} g.$$

Then the commutativity of (5.2.11) implies firstly that

$$\begin{aligned}(\mathrm{Nrd}_{\mathbb{C}_p[G]}(\langle \theta, \iota_1, \iota_2\rangle))_{1_L^F} &= (\mathrm{Nrd}_{\mathbb{C}_p[G(L)]}(\langle \theta, \iota_1, \iota_2\rangle))_{1_L} \\ &= \det_{\mathbb{C}_p}\big(\langle \theta, \iota_1, \iota_2\rangle_{G(L)}\big),\end{aligned} \tag{5.2.12}$$

where $\langle \theta, \iota_1, \iota_2\rangle_{G(L)}$ is the $\mathbb{C}_p$-automorphism of $e_{G(L)}(\mathbb{C}_p \cdot P)$ induced by $\langle \theta, \iota_1, \iota_2\rangle$.

Next we note that the element $v$ in Lemma 5.9 belongs to $\xi(\mathbb{Z}_p[G])^\times$ and thus to the unit group of the (unique) maximal $\mathbb{Z}_p$-order in $\zeta(\mathbb{Q}_p[G])$. Hence, if $\mathcal{M}$ is any maximal $\mathbb{Z}_p$-order in $\mathbb{Q}_p[G]$ as in Remark 5.10, then [19, Prop. (45.8)] implies the existence of an element $x$ of $\mathcal{M}^\times$ with $v = \mathrm{Nrd}_{\mathbb{C}_p[G]}(x)$. From the commutativity of the diagram (5.2.11), we can therefore deduce that

$$v_{1_L^F} = (\mathrm{Nrd}_{\mathbb{C}_p[G]}(x))_{1_L^F} = (\mathrm{Nrd}_{\mathbb{C}_p[G(L)]}(x))_{1_L} = \mathrm{Nrd}_{\mathbb{C}_p}(x_{G(L)}),$$

where $x_{G(L)}$ is the automorphism of $\mathbb{C}_p[G(L)]e_{G(L)}$ induced by (right) multiplication by $x$.

In particular, since $x_{G(L)}$ restricts to give an automorphism of the free rank one $\mathbb{Z}_p$-module $\mathcal{M}e_{G(L)}$, the last displayed equalities imply that $v_{1_L^F}$ belongs to $\mathbb{Z}_p^\times$. Given this fact (and that $p$ is assumed to be odd), the final assertion of Theorem B(iii) follows upon substituting into the equality (5.2.9) each of (5.2.10), (5.2.12) and the containment that is established in the following result.

**Lemma 5.11.** *Let $L$ be a subfield of $F$ and write $R_{L,S}$ for its $S$-regulator. Then one has*

$$\det_{\mathbb{C}_p}\big(\langle \theta, \iota_1, \iota_2\rangle_{G(L)}\big) \cdot \big(R_{L,S} \cdot |\mathrm{Cl}(\mathcal{O}_{L,S})|\big)^{-1} \in \mathbb{Z}_p^\times.$$

*Proof.* We set $J := G(L)$ and for each $G$-module $M$ abbreviate the modules of $J$-invariants $H^0(J, M)$ and $J$-coinvariants $H_0(J, M)$ to $M^J$ and $M_J$ respectively.

Then, since the functors $M \to M^J$ and $M \mapsto M_J$ are respectively left and right exact, the exact sequence (5.2.6) gives rise to an exact commutative diagram of $\mathbb{Z}_p[G/J]$-modules

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_{L,S,p}^\times \arrow[r, "\iota^J"] & P^J \arrow[r, "\theta^J"] & P^J & & \\ & & P_J \arrow[u, "\cong"'] \arrow[r, "\theta_J"] & P_J \arrow[u, "\cong"] \arrow[r, "\varpi"] & H^1(C_{L,S}(\mathbb{Z}_p(1))) \arrow[r] & 0. \end{tikzcd}$$

Here the two vertical maps are induced by sending each element $x$ of $P$ to $\sum_{g \in J} g(x)$ and so are bijective since $P$ is a free $J$-module, and $\varpi$ denotes the composite of $\pi_J$ and the isomorphism $H^1(C_F)_J \cong H^1(C_{L,S}(\mathbb{Z}_p(1)))$ induced by the relevant case of Lemma 4.1(iv).

We also recall (from [47, Chap. I, §6.5]) the commutative diagram of $\mathbb{R}[G]$-modules

$$\begin{array}{ccc} \mathbb{R}\cdot\mathcal{O}^\times_{F,S} & \xrightarrow{\mathrm{Reg}_F} & \mathbb{R}\cdot X_{F,S} \\ \cup\big\uparrow & & \big\uparrow \\ \mathbb{R}\cdot\mathcal{O}^\times_{L,S} & \xrightarrow{\mathrm{Reg}_L} & \mathbb{R}\cdot X_{L,S} \end{array}$$

in which the right hand vertical homomorphism is induced by sending each place $v$ of $S_L$ to $\sum_{g\in J} g(w)$ where $w$ is any fixed place of $F$ lying above $v$.

In particular, the above two diagrams combine to imply that $\langle\theta,\iota_1,\iota_2\rangle_J$ identifies with the automorphism of

$$\mathbb{C}_p\cdot P^J = (\mathbb{C}_p\cdot P)^J = (\mathbb{C}_p\cdot\mathcal{O}^\times_{L,S,p})\oplus(\mathbb{C}_p\otimes_{\mathbb{Q}_p}\iota_1)(\mathbb{C}_p\cdot\mathrm{im}(\theta^J))$$

that is equal to the composite $(\mathbb{C}_p\otimes_{\mathbb{Q}_p}\iota_2^J)\circ(\mathbb{C}_p\otimes_{\mathbb{R}}\mathrm{Reg}_L)$ on $\mathbb{C}_p\cdot\mathcal{O}^\times_{L,S,p}$ and to $\mathbb{C}_p\otimes_{\mathbb{Z}_p}\theta^J$ on $(\mathbb{C}_p\otimes_{\mathbb{Q}_p}\iota_1)(\mathbb{C}_p\cdot\mathrm{im}(\theta^J))$.

In addition, if we write $P^\bullet$ for the complex $P^\bullet_F$ that is considered in Lemma 4.10(iii) (with $T=\mathbb{Z}_p(1)$), then Lemma 4.1(iv) implies that the complex $(P^\bullet)^J\cong(P^\bullet)_J$ identifies with $C_{L,S}(\mathbb{Z}_p(1))$ in such a way that $\mathbb{C}_p\otimes_{\mathbb{Q}_p}\iota_2^J$ is a section to the composite homomorphism

$$\begin{aligned}\mathbb{C}_p\cdot P^J \cong \mathbb{C}_p\cdot P_J &\to \mathbb{C}_p\cdot H^1((P^\bullet)_J)\\ &= \mathbb{C}_p\cdot H^1(C_{L,S}(\mathbb{Z}_p(1)))\\ &\cong \mathbb{C}_p\cdot X_{L,S,p}\cong\mathbb{C}_p\cdot X^J\end{aligned}$$

in which the arrow denotes the tautological map, the second isomorphism is induced by the exact sequence in Remark 5.2 (with $F$ and $\Sigma$ replaced by $L$ and $S$) and the final isomorphism is induced by the right hand vertical map in the diagram above.

Given this explicit description of $\langle\theta,\iota_1,\iota_2\rangle_J$, and the description of the cohomology groups of $C_{L,S}(\mathbb{Z}_p(1))$ in Remark 5.2, the claimed result is verified by an explicit computation of $\det_{\mathbb{C}_p}\big(\langle\theta,\iota_1,\iota_2\rangle_J\big)$ using the methods, for example, of [5, Lem. A.1 and Lem. A.3]. However, since this computation is routine we leave details to the reader. □

This completes the proof of Theorem B.

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